

Lévy–Schrödinger wave packets

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Abstract

We analyze the time-dependent solutions of the pseudo-differential Lévy–Schrödinger wave equation in the free case, and we compare them with the associated Lévy processes. We list the principal laws used to describe the time evolutions of both the Lévy process densities, and the Lévy–Schrödinger wave packets. To have self-adjoint generators and unitary evolutions we will consider only absolutely continuous, infinitely divisible Lévy noises with laws symmetric under change of sign of the independent variable. We then show several examples of the characteristic behavior of the Lévy–Schrödinger wave packets, and in particular of the bi-modality arising in their evolutions: a feature at variance with the typical diffusive uni-modality of both the Lévy process densities, and the usual Schrödinger wave functions.

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1 Introduction and notations

In a recent paper [1] it has been shown how to extend the well known relation between the Wiener process and the Schrödinger equation [2, 3, 4, 5] to other suitable Lévy process. This idea – discussed elsewhere only in the stable case [6, 7] – leads to a *L-S* (Lévy–Schrödinger) equation containing additional integral terms which take into account the possible jumping part of the background noise. In fact, the infinitesimal generator of the Brownian semigroup (the Laplacian) being substituted by the more general generator of a Lévy semigroup, we get an integro-differential operator with both a continuous (differential and Gaussian) and a jumping (integral, non Gaussian) part. These ideas have already been discussed in the framework of *stochastic mechanics* [2, 5] and are considered as a model for systems more general than just the usual quantum mechanics: a true *dynamical theory of Lévy processes*

that can be applied to several physical problems [8]. The aim of this paper is now to show a number of explicit examples of wave packets solutions of L - S free equations.

In recent years we have witnessed a considerable growth of interest in non Gaussian stochastic processes – and in particular into Lévy processes – from statistical mechanics to mathematical finance. In the physical field, however, the research scope is presently rather confined to the stable processes and to the corresponding fractional calculus [6, 7, 9], while in the financial domain a vastly more general type of processes is at present in use. Here we suggest that a Lévy stochastic mechanics should be considered as a dynamical theory of the entire gamut of the *infinitely divisible* processes with time reversal invariance, and that the horizon of its applications should be widened even to cases different from the quantum systems.

This approach has several advantages: first of all the use of general infinitely divisible processes lends the possibility of having realistic, finite variances. Second, the presence of a Gaussian component and the wide spectrum of decay velocities of the increment densities will give the possibility of having models with differences from the usual Brownian (and usual quantum mechanical, Schrödinger) case as small as we want. Last but not least, there are examples of non stable Lévy processes which are connected with the simplest form of the quantum, *relativistic* Schrödinger equation: an important link that was missing in the original Nelson model. This final remark, on the other hand, shows that the present inquiry is not only justified by the a desire of formal generalization, but is required by the need to attain physically meaningful cases that otherwise would not be contemplated in the narrower precinct of the stable laws.

In this paper we will show practical examples for the behavior of the evolving wave packet solutions of particular kinds of (non Wiener) L - S equations, and we will put in evidence their characteristics. In particular the *bi-modality* arising in many of these these evolutions which has a correspondence neither in the the process diffusions, nor in the usual Schrödinger wave functions: an effect which has already been observed only in confined Lévy flights [10]. This is coherent with the usual stochastic mechanics scheme, in so far as in this theory the Schrödinger equation is recovered by introducing a kind of interaction modeled by means of a *quantum potential* [2, 5]. In the following exposition laws and processes will always be one dimensional. An extensive analysis of the topics discussed in this first chapter is available in the two monographs [11] and [12], while a short introduction can be found in [13].

In the present paper the law of a *rv* (random variable) X is characterized either by its *pdf* (probability density function) f , when – as it is generally supposed – the law is *ac* (absolutely continuous), or by its *chf* (characteristic function) φ with the usual reciprocity relations

$$\varphi(u) = \int_{-\infty}^{+\infty} f(x)e^{iux} dx, \quad f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(u)e^{-iux} du. \quad (1)$$

When the laws are not *ac* we sometimes will use the *Dirac delta* notation: the

symbol $\delta_{x_0}(x) = \delta(x - x_0)$ will then represent a law degenerated in x_0 and only formally it will act as a *pdf*. The symbol $\delta(x)$ will also be used instead of $\delta_0(x)$. In order to have background noises with generators self-adjoint in L^2 – an essential requirement for our purposes – we will consider only *symmetric* laws, namely we will require

$$f(-x) = f(x), \quad \varphi(-u) = \varphi(u)$$

so that the *chf* φ will also be real. This also means that, when it exists, the expectation vanishes $\mathbf{E}[X] = 0$, namely the law is also *centered*. See the Appendix A for further details about our notations.

Since we will restrict our analysis to background noises driven by Lévy processes, we will be interested almost exclusively in *id* (infinitely divisible¹, for details see [11, 12, 13]) laws with a Lévy triplet $\mathcal{L} = (\alpha, \beta, \nu)$. Here our Lévy measures ν will always be supposed to have a density: $\nu(dy) = \ell(y) dy$; when this does not happen we will often use the Dirac delta notation. As a consequence the Lévy triplet will be rather specified as $\mathcal{L} = (\alpha, \beta, \ell)$. The *lch* (logarithmic characteristic) of our *id* laws $\eta = \ln \varphi$, with $\varphi = e^\eta$, will then satisfy the Lévy–Khintchin formula

$$\eta(u) = i\alpha u - \frac{1}{2}\beta^2 u^2 + \int_{y \neq 0} [e^{iuy} - 1 - iuy I_D(y)] \ell(y) dy \quad (2)$$

where $D = \{y : |y| < 1\}$. The prescription of the integral around the origin is essential only when – as usually may happen – the Lévy measure shows a singularity in $y = 0$. When the law is dimensionless (see Appendix A) then also α, β, ℓ and y are so; on the other hand, if the law has the dimensions of a length, then α, β, y are lengths, while ℓ is the reciprocal of a length. In particular when the law is symmetric we have

$$\alpha = 0, \quad \ell(-x) = \ell(x)$$

so that the Lévy–Khintchin formula will be reduced to the symmetric real expression

$$\eta(u) = -\frac{1}{2}\beta^2 u^2 + \int_{y \neq 0} (\cos uy - 1) \ell(y) dy \quad (3)$$

and hence the *chf* φ will not only be real, but also non negative: $\varphi(u) \geq 0$.

The Markov processes dealt with in this paper are stationary, independent increments processes and are then defined by means of the *chf* $\varphi^{\Delta t/\tau}$ of their Δt -increments, where τ is a dimensional, time scale parameter. Here too we can introduce a dimensionless formulation through the coordinate t/τ , and to simplify the notation we can continue to use the same symbol t for this dimensionless time. In this case the stationary *chf* will be reduced to $\varphi^{\Delta t}$, and the dimensional formulation

¹A law φ is said to be *id* if for every n it exists a *chf* φ_n such that $\varphi = \varphi_n^n$; on the other hand φ is said to be stable when for every $c > 0$ it is always possible to find $a > 0$ and $b \in \mathbf{R}$ such that $e^{ibu} \varphi(au) = [\varphi(u)]^c$. Every stable law is also *id*. See also Appendix A for further details about stable laws.

will be recovered by simple substitution of t/τ to t . A stochastically continuous process with stationary and independent increments is called a *Lévy process* when $X(0) = 0$, \mathbf{P} -a.s., but this paper will mostly be about the same kind of processes for arbitrary initial conditions $X(0) = X_0$, \mathbf{P} -a.s. with law $f_0(x)$ and $\varphi_0(u) = e^{\eta_0(u)}$. All these processes, independently from their initial conditions, will share both the same differential equations (whether *SDE*'s, or *PDE*'s) and the same transition *pdf*'s

$$f_{X(t)}(x | X(s) = y) = p(x, t | y, s).$$

To avoid confusion we will then adopt different notations for their respective marginal *pdf*'s: for a Lévy process (namely with $X_0 = 0$ initial condition) we will write

$$f_{X(t)}(x) = q(x, t), \quad \varphi_{X(t)}(u) = \chi(u, t)$$

with $q(x, 0) = \delta(x)$ and $\chi(x, 0) = 1$, while for the general stationary and independent increments process (with arbitrary initial condition X_0) we will write

$$f_{X(t)}(x) = p(x, t), \quad \varphi_{X(t)}(u) = \phi(u, t)$$

with $p(x, 0) = f_0(x)$ and $\phi(x, 0) = \varphi_0(x)$. It is then easy to show that

$$p(x, t | y, s) = q(x - y, t - s). \quad (4)$$

The infinitesimal generator $A = \eta(\partial)$ (here ∂ stands for the derivation with respect to the variable of a test function v) of the semigroup of a Lévy process will be a pseudo-differential operator with symbol η [1, 12], namely from (2)

$$\begin{aligned} [Av](x) &= [\eta(\partial)v](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{iux} \eta(u) \hat{v}(u) du \\ &= \alpha \partial_x v(x) + \frac{\beta^2}{2} \partial_x^2 v(x) \\ &\quad + \int_{y \neq 0} [v(x+y) - v(x) - y I_D(y) \partial_x v(x)] \ell(y) dy \end{aligned} \quad (5)$$

where \hat{v} denotes the *FT* (Fourier transform) of the test function v with the reciprocity relations:

$$\hat{v}(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} v(x) e^{-iux} dx, \quad v(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{v}(u) e^{iux} du$$

The generator A will be self-adjoint in $L^2(\mathbb{R}, dx)$ when the law is symmetric, and in this case it reduces to

$$[Av](x) = \frac{\beta^2}{2} \partial_x^2 v(x) + \int_{y \neq 0} [v(x+y) - v(x)] \ell(y) dy \quad (6)$$

<i>law</i>	<i>f</i>	φ	β	ℓ	E	V
\mathfrak{D}	$\delta(x)$	1	0	0	0	0
\mathfrak{N}	$\frac{e^{-x^2/2}}{\sqrt{2\pi}}$	$e^{-u^2/2}$	1	0	0	1
\mathfrak{C}	$\frac{1}{\pi} \frac{1}{1+x^2}$	$e^{- u }$	0	$\frac{1}{\pi x^2}$	–	$+\infty$
\mathfrak{L}	$\frac{e^{- x }}{2}$	$\frac{1}{1+u^2}$	0	$\frac{e^{- x }}{ x }$	0	2
\mathfrak{U}	$\frac{\Theta(x+1)-\Theta(x-1)}{2}$	$\frac{\sin u}{u}$	–	–	0	$\frac{1}{3}$
\mathfrak{D}_1	$\frac{\delta_1(x)+\delta_{-1}(x)}{2}$	$\cos u$	–	–	0	1

Table 1: List of the essential properties of a few basic, dimensionless laws discussed in this paper: degenerate (Dirac) \mathfrak{D} , normal (Gauss) \mathfrak{N} , Cauchy \mathfrak{C} , Laplace \mathfrak{L} , uniform \mathfrak{U} , and doubly degenerate in $+1, -1$ (symmetric Bernoulli) \mathfrak{D}_1 .

so that it is determined by the two essential elements of our Lévy triplet, namely β and ℓ . Given the process stationarity, in a dimensionless formulation the transition law degenerate in $x = 0$ at $t = 0$ will have as *chf* $\chi = \varphi^t = e^{t\eta}$ and as *pdf*

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(u, t) e^{-iux} du = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi(u)^t e^{-iux} du. \quad (7)$$

This transition law plays an important role in the evolution of an arbitrary initial law f_0 , φ_0 : the process *chf* will indeed be now $\phi(u, t) = \chi(u, t)\varphi_0(u)$, and the corresponding *pdf* will be calculated from

$$p(x, t) = [q(t) * f_0](x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(u, t) e^{-iux} du$$

namely either as a convolution of the transition and the initial *pdf*'s, or by inverting the *chf* ϕ of the process. This *pdf* will also be a solution of the evolution pseudo-differential equation [1, 12]

$$\partial_t p = \eta(\partial)p, \quad p(x, 0) = f_0(x) \quad (8)$$

which from (5) takes the integro-differential form

$$\begin{aligned} \partial_t p(x, t) &= \alpha \partial_x p(x, t) + \frac{\beta^2}{2} \partial_x^2 p(x, t) \\ &\quad + \int_{y \neq 0} [p(x + y, t) - p(x, t) - y I_D(y) \partial_x p(x, t)] \ell(y) dy \end{aligned}$$

<i>law</i>	$[Av](x)$
\mathfrak{N}	$\frac{\partial_x^2 v(x)}{2}$
\mathfrak{C}	$\int_{y \neq 0} \frac{v(x+y)-v(x)}{\pi y^2} dy$
\mathfrak{L}	$\int_{y \neq 0} \frac{[v(x+y)-v(x)]e^{- y }}{ y } dy$

Table 2: List of the generators of the Lévy processes associated to some of the non degenerate, *id*, dimensionless laws of Table 1

and for a centered, symmetric noise from (6) reduces to

$$\partial_t p(x, t) = \frac{\beta^2}{2} \partial_x^2 p(x, t) + \int_{y \neq 0} [p(x + y, t) - p(x, t)] \ell(y) dy. \quad (9)$$

We finally remember that, since (8) and (9) are given in terms of process *pdf*'s, this equations are supposed to hold only for *ac* processes. We are then required to point out which Lévy processes have densities. To answer – at least partially – this question we then recall that [11] any non-degenerate, *sd* (self-decomposable², for details see [11, 12, 13]) distribution is *ac*. On the other hand such a property also extends to the corresponding processes for every t . In fact [11] if $X(t)$ is a *sd* process also its *pdf* at every t is *sd*, and hence $X(t)$ is *ac* for every t . As a consequence we can always explicitly write down the evolution equations (9) in terms of the process *pdf*'s at least for the *sd* case. We remark, however, that there are also non *sd* processes which are *ac*: the *ac* compound Poisson processes of Appendix B are an example in point.

We listed in the Table 1 the properties of a few basic, symmetric, dimensionless laws: degenerate (Dirac) \mathfrak{D} , normal (Gauss) \mathfrak{N} , Cauchy \mathfrak{C} , Laplace \mathfrak{L} , uniform \mathfrak{U} , and doubly degenerate in $+1, -1$ (symmetric Bernoulli) \mathfrak{D}_1 . The uniform law *pdf* is given by means of the Heaviside functions $\Theta(x)$. These laws are also relevant particular cases of the families that we will introduce in the Section 2. Remark that in the Table 1 there is no value for the expectation of \mathfrak{C} because it does not exist (\mathfrak{C} is centered on the median), and no values for the Lévy triplet of \mathfrak{U} and \mathfrak{D}_1 since these are not *id* laws. Moreover in general our laws are not necessarily standard. The form of the simplest generators corresponding to our Lévy processes is finally shown in the Table 2.

The paper is organized as follows: in the Chapter 2 we recall the essential properties of the law families of our interest; then in the Chapter 3 the *L-S* equation is introduced with its connections to the Lévy processes. Finally in Chapter 4 our

²A law $\varphi(u)$ is *sd* when for every $a \in (0, 1)$ we can always find another *chf* $\varphi_a(u)$ such that $\varphi(u) = \varphi(au)\varphi_a(u)$. Every stable law is also *sd*; every *sd* law is also *id*.

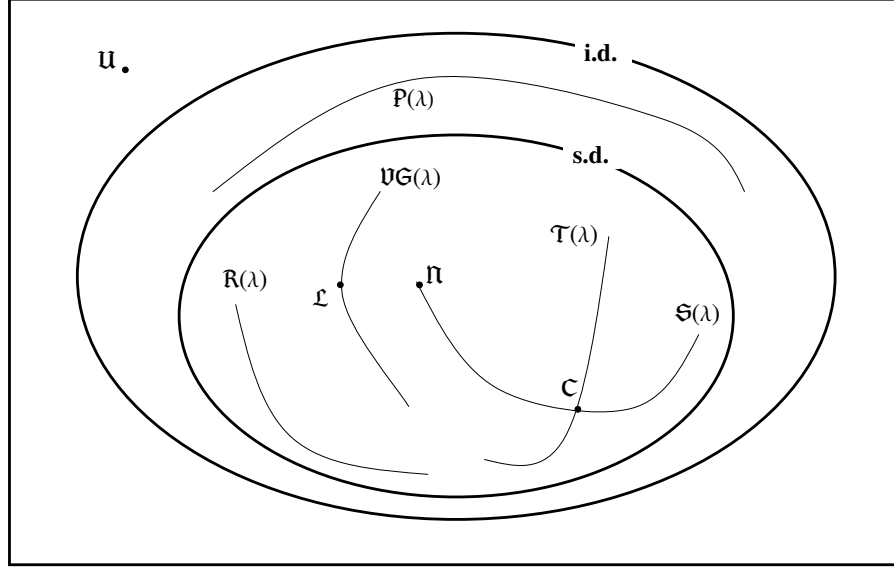


Figure 1: Graphical synthesis of the relations among the families of laws discussed in the Section 2. The uniform \mathcal{U} is our unique example beyond the pale of the *id* laws, while the laws of the (simple) Poisson family $\mathcal{P}(\lambda)$ are *id* but not *sd*. Notable cases (\mathcal{N} , \mathcal{L} , \mathcal{C}) within the *sd* families are put in evidence; the Cauchy \mathcal{C} law lies at the intersection of the stable $\mathcal{S}(\lambda)$ and Student $\mathcal{T}(\lambda)$ families.

examples are elaborated and in Chapter 5 the results are collected and discussed. A few technical details are collected in the Appendices in order to avoid to excessively burden the text.

2 Families of *id* laws

We will introduce here the principal families of *id* laws considered in this paper. For a graphical synthesis of the relations among them see Figure 1. Please remark that this synthesis is particularly simple because we limit ourselves here to *dimensionless* laws (see Appendix A): this produces one-parameter families that can be easily represented in our scheme. In the Table 3 are then listed the properties of the principal families of dimensionless, *sd* laws that will be discussed. The “...” symbol in this table means either that we do not have an elementary formulation for the entry, or that there are no particular values of λ to be put in evidence. K_ν , B and Γ respectively are the modified Bessel functions of the second kind, and the Euler Beta and Gamma functions, while H_λ stands for the Fox H -functions representing the *pdf* of stable laws [14]. From Table 1 and Table 3 we can on the other hand immediately see that $\mathcal{S}(1) = \mathcal{T}(1) = \mathcal{C}$, $\mathcal{S}(2) = \mathcal{N}$ and $\mathcal{V}\mathcal{G}(1) = \mathcal{L}$, as also put in evidence in the Figure 1. The behaviors of a few *lch*’s of *id* laws are finally displayed and compared in the Figure 2: it could be seen there that all the *lch*’s of the *sd*

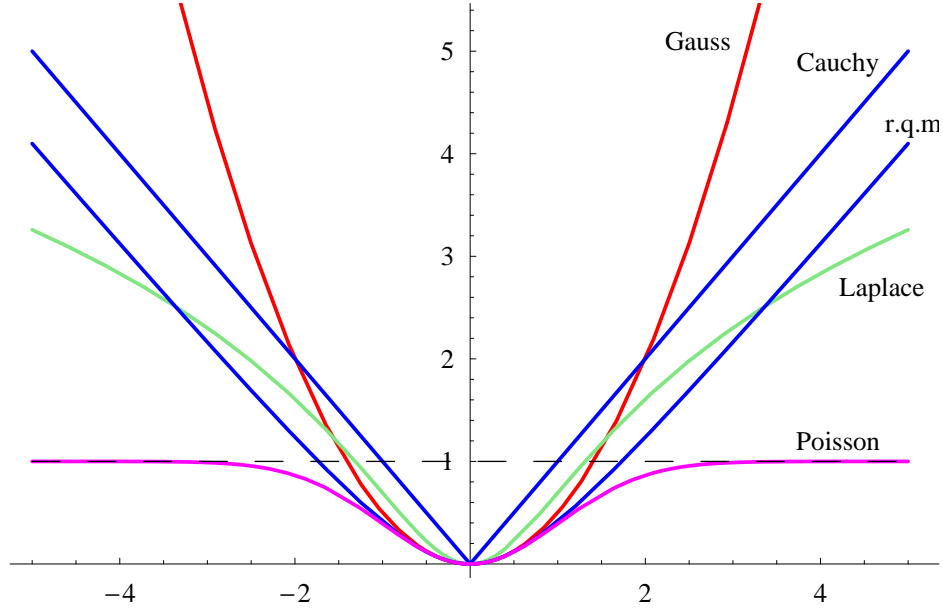


Figure 2: The $lch -\eta(u)$ of some basic dimensionless laws from Table 3, plus that of a compound Poisson $\mathfrak{P}(\lambda, \mathfrak{N})$ with normal component laws (see Appendix B).

law	f	φ	β	ℓ	$\lambda > 0$
$\mathfrak{S}(\lambda)$	$H_\lambda(x)$	$e^{- u ^\lambda/\lambda}$	0	$\frac{ x ^{-1-\lambda}}{-2\lambda\Gamma(-\lambda)\cos(\lambda\pi/2)}$	< 2
	$\frac{1}{\pi} \frac{1}{1+x^2}$	$e^{- u }$	0	$\frac{1}{\pi x^2}$	1
	$\frac{e^{-x^2/2}}{\sqrt{2\pi}}$	$e^{-u^2/2}$	1	0	2
$\mathfrak{VG}(\lambda)$	$\frac{ x ^{\lambda-1/2}K_{\lambda-1/2}(x)}{2^{\lambda-1}\Gamma(\lambda)\sqrt{2\pi}}$	$\left(\frac{1}{1+u^2}\right)^\lambda$	0	$\frac{\lambda e^{- x }}{ x }$	\dots
	$\frac{e^{- x }}{2}$	$\frac{1}{1+u^2}$	0	$\frac{e^{- x }}{ x }$	1
$\mathfrak{T}(\lambda)$	$\frac{1}{B(\frac{1}{2}, \frac{\lambda}{2})} \left(\frac{1}{1+x^2}\right)^{\frac{\lambda+1}{2}}$	$\frac{2 u ^{\lambda/2}K_{\lambda/2}(u)}{2^{\lambda/2}\Gamma(\lambda/2)}$	0	\dots	\dots
	$\frac{1}{\pi} \frac{1}{1+x^2}$	$e^{- u }$	0	$\frac{1}{\pi x^2}$	1
$\mathfrak{R}(\lambda)$	$\frac{\lambda e^\lambda K_1(\sqrt{\lambda^2+x^2})}{\pi\sqrt{\lambda^2+x^2}}$	$e^{\lambda(1-\sqrt{1+u^2})}$	0	$\frac{\lambda K_1(x)}{\pi x }$	\dots

Table 3: Properties of our principal families of sd , dimensionless laws: the stable $\mathfrak{S}(\lambda)$, the Variance–Gamma $\mathfrak{VG}(\lambda)$, the Student $\mathfrak{T}(\lambda)$ and the relativistic qm (quantum mechanics) $\mathfrak{R}(\lambda)$.

laws considered in this paper diverge at infinity with velocities ranging from u^2 to $\log u$, while the unique not diverging *lch* characterizes one of our non *sd* examples: the compound Poisson $\mathfrak{P}(\lambda, \mathfrak{N})$ with normal component laws. This also gives an intuitive idea of how much the behavior of a law – in so far as we are concerned, for instance, with its jumping properties – differs from that of the \mathbf{P} -a.s. continuous Gaussian case.

2.1 The stable laws $\mathfrak{S}(\lambda)$

This is the more widely studied family of *id* laws, albeit among them only the normal $\mathfrak{S}(2) = \mathfrak{N}$ enjoys a finite variance. But for the \mathfrak{N} , the \mathfrak{C} and precious few other cases the *pdf*'s of the stable laws exist only in the form of Fox *H*-functions [14]. To see in what sense these laws are *stable* we must for a moment reintroduce the dimensional parameter a : we then have a larger family $\mathfrak{S}_a(\lambda)$ with two parameters, $0 < \lambda \leq 2$ and $a > 0$, and

$$\varphi(u) = e^{-a^\lambda |u|^\lambda / \lambda}. \quad (10)$$

Now, for a given fixed λ , the family $\mathfrak{S}_a(\lambda)$ with $a > 0$ is closed under convolution, as can be easily seen from (10). For instance the families of the normal $\mathfrak{N}_a = \mathfrak{N}(a^2)$ and Cauchy \mathfrak{C}_a laws are closed under convolution since $\mathfrak{N}(a_1^2) * \mathfrak{N}(a_2^2) = \mathfrak{N}(a_1^2 + a_2^2)$ and $\mathfrak{C}_{a_1} * \mathfrak{C}_{a_2} = \mathfrak{C}_{a_1 + a_2}$. Stability however means more: the families $\mathfrak{S}_a(\lambda)$ for a given λ are *types* of laws, in the sense that a law of the family differs from another just by a re-scaling (centering is not necessary here because our laws already are centered; for details see Appendix A), the parameter a being indeed nothing else than a space scale parameter. This has far reaching consequences. In particular it is at the root of the well known fact that the stable Lévy processes are *self-similar*: a property not extended to other, non stable Lévy processes [13]. The generators of the stable Lévy processes are

$$[Av](x) = \frac{-1}{2\lambda\Gamma(-\lambda)\cos\frac{\lambda\pi}{2}} \int_{y \neq 0} \frac{v(x+y) - v(x)}{|y|^{1+\lambda}} dy \quad 0 < \lambda < 2, \quad \lambda \neq 1$$

while for $\lambda = 1$ (\mathfrak{C} law) and $\lambda = 2$ (\mathfrak{N} law) they are listed in the Table 2.

2.2 The Variance–Gamma laws $\mathfrak{VG}(\lambda)$

The Variance–Gamma laws owe their name to the fact that they can be seen as *normal variance-mean mixtures*³ where the mixing density is a *gamma* distribution.

³A normal variance-mean mixture, with mixing probability density g , is the law of a random variable Y of the form $Y = \alpha + \beta V + \sigma\sqrt{V}X$ where α and β are real numbers and $\sigma > 0$. The random variables X and V are independent; X is a normal standard, and V has a *pdf* g with support on the positive half-axis. The conditional distribution of Y given V is then a normal distribution with mean $\alpha + \beta V$ and variance $\sigma^2 V$. A normal variance-mean mixture can be thought of as the distribution of a certain quantity in an inhomogeneous population consisting of many different normally distributed sub-populations.

It is apparent moreover from the Table 3 that $\mathfrak{VG}(\lambda)$ is closed under convolution in the sense that $\mathfrak{VG}(\lambda_1) * \mathfrak{VG}(\lambda_2) = \mathfrak{VG}(\lambda_1 + \lambda_2)$. That notwithstanding, however, the Variance–Gamma laws are not stable. To see that let us reintroduce the dimensional scale parameter a to have the enlarged family $\mathfrak{VG}_a(\lambda)$:

$$\varphi(u) = \left(\frac{1}{1 + a^2 u^2} \right)^\lambda$$

Now every sub-family with a given, fixed a is closed under convolution, but at variance with the stable case the parameter describing the sub-family is λ , rather than a . As a consequence the closed subfamilies do not constitute types of laws differing only by a rescaling, and hence the laws are not stable. The *pdf*'s of the Variance–Gamma laws can be given in particular instances as finite combinations of elementary functions. By generalizing the quoted example of the Laplace law $\mathfrak{VG}(1) = \mathfrak{L}$, when $\lambda = n + 1$ with $n = 0, 1, \dots$ we have for the dimensionless *pdf*'s

$$f(x) = \sum_{k=0}^n \binom{2n-k}{n} \frac{(2|x|)^k e^{-|x|}}{k! 2^{2n+1}} = \frac{e^{-|x|}}{n! 2^{n+1}} \theta_n(|x|)$$

where $\theta_n(x)$ are reverse Bessel polynomials [15]. All our dimensionless $\mathfrak{VG}(\lambda)$ laws are endowed with expectations (which vanish by symmetry) and finite variances 2λ . The generator of the corresponding Lévy process is

$$[Av](x) = \lambda \int_{y \neq 0} \frac{v(x+y) - v(x)}{|y|} e^{-|y|} dy \quad \lambda > 0$$

which coincides with that of \mathfrak{L} (see Table 2) for $\lambda = 1$.

2.3 The Student laws $\mathfrak{T}(\lambda)$

But for the Cauchy \mathfrak{C} case, the laws of the Student family (even enlarged by means of the scale parameter a) are not stable, and $\mathfrak{T}(\lambda)$ itself is not closed under convolution: convolutions of Student laws are not Student laws. As can be seen from Table 3 the Variance–Gamma and the Student families enjoy a sort of duality since their *pdf*'s and *chf*'s are essentially exchanged. This has been discussed at length in a few recent papers [17, 18, 19]. Remark that to put in evidence this correspondence we have chosen the Student laws of $\mathfrak{T}(\lambda)$ without introducing the usual parametric scaling x^2/λ of its variable that would have put equal to $\lambda/(\lambda-2)$ all their variances for $\lambda > 2$. In particular this means that for $\lambda \rightarrow +\infty$ we will not get a standard \mathfrak{N} law, as also shown in the Figure 1. The following remarks are however virtually untouched by this choice. While the *pdf*'s and *chf*'s of the Student laws are known, differently from the Variance–Gamma laws their Lévy measures and generators have not a known general expression. However we can give them in particular instances. For example when $\lambda = 2n + 1$ with $n = 0, 1, \dots$ the *chf* becomes

$$\varphi(u) = \sum_{k=0}^n \frac{n!(2n-k)!}{(2n)!(n-k)!k!} (2|u|)^k e^{-|u|} = \frac{n! 2^n e^{-|u|}}{(2n)!} \theta_n(|u|).$$

where θ_n are again reverse Bessel polynomials [15]. Of course $\mathfrak{T}(1) = \mathfrak{C}$ is the well known Cauchy (stable) case, while for $\mathfrak{T}(3)$ we have

$$f(x) = \frac{2}{\pi} \left(\frac{1}{1+x^2} \right)^2, \quad \varphi(u) = (1+|u|)e^{-|u|}.$$

and it can be shown [17] in this case that the Lévy measure is

$$\ell(x) = \frac{1 - |x|(\sin |x| \operatorname{ci} |x| - \cos |x| \operatorname{si} |x|)}{\pi x^2} \quad (11)$$

where the sine and the cosine integral functions are

$$\operatorname{si} x = - \int_x^{+\infty} \frac{\sin y}{y} dy, \quad \operatorname{ci} x = - \int_x^{+\infty} \frac{\cos y}{y} dy$$

The existence of the moments of the $\mathfrak{T}(\lambda)$ laws depends on the value of the parameter λ : the n^{th} moment exists if $n < \lambda$. In particular the expectation exists (and vanishes) for $\lambda > 1$, while the variance exists finite for $\lambda > 2$ and its value is $(\lambda - 2)^{-1}$. The generator of the Lévy process can finally be explicitly given for $\mathfrak{T}(3)$ from (11)

$$[Av](x) = \lambda \int_{y \neq 0} [v(x+y) - v(x)] \frac{1 - |y|(\sin |y| \operatorname{ci} |y| - \cos |y| \operatorname{si} |y|)}{\pi y^2} dy.$$

2.4 The compound Poisson laws $\mathfrak{N}_\sigma * \mathfrak{P}(\lambda, \mathfrak{H})$

The compound Poisson laws $\mathfrak{H}_0 * \mathfrak{P}(\lambda, \mathfrak{H})$ are not *sd*, but they are nevertheless *id*; they are also *ac* when \mathfrak{H}_0 is *ac* (for details and notations see Appendix B). In the following examples we will take into account the dimensional parameters a of the component laws. Consider now the case $\mathfrak{H}_0 = \mathfrak{N}_\sigma$: then $\ell_0(x) = 0$ and $\beta_0 = \sigma$ so that the Lévy triplet of $\mathfrak{N}_\sigma * \mathfrak{P}(\lambda, \mathfrak{H})$ is $\mathcal{L} = (0, \sigma, \lambda h)$ and the generator is

$$[Av](x) = \frac{\sigma^2}{2} \partial_x^2 v(x) + \lambda \int_{y \neq 0} [v(x+y) - v(x)] h(y) dy.$$

When in particular also $\mathfrak{H} = \mathfrak{N}_a$, then the Lévy triplet of $\mathfrak{N}_\sigma * \mathfrak{P}(\lambda, \mathfrak{N}_a)$ is

$$\mathcal{L} = \left(0, \sigma, \lambda \frac{e^{-x^2/2a^2}}{\sqrt{2\pi a^2}} \right)$$

and we get a law with the following *pdf* and *lch*

$$f(x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{e^{-x^2/2(ka^2+\sigma^2)}}{\sqrt{2\pi(ka^2+\sigma^2)}}, \quad \eta(u) = \lambda(e^{-a^2 u^2/2} - 1) - \frac{\sigma^2 u^2}{2},$$

namely a Poisson mixture of centered normal laws $\mathfrak{N}(ka^2 + \sigma^2)$. The self-adjoint generator then is

$$[Av](x) = \frac{\sigma^2}{2} \partial_x^2 v(x) + \lambda \int_{-\infty}^{+\infty} [v(x+y) - v(x)] \frac{e^{-y^2/2a^2}}{\sqrt{2\pi a^2}} dy.$$

and we could look at it as to a Poisson correction to the Wiener generator, the relative weight of these two independent components being ruled by the ratio between λ and σ^2 .

As another example of *ac* compound Poisson law let us suppose instead that $\mathfrak{H}_0 = \mathfrak{N}_\sigma$ again, but that $\mathfrak{H} = \mathfrak{D}_a$ (see Appendix B), so that the Lévy triplet of $\mathfrak{N}_\sigma * \mathfrak{P}(\lambda, \mathfrak{D}_a)$ now is

$$\mathcal{L} = \left(0, \sigma, \lambda \frac{\delta_1(x/a) + \delta_{-1}(x/a)}{2a} \right)$$

while its *pdf* and *lch* are

$$\begin{aligned} f(x) &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} \frac{e^{-[x-(k-2j)a]^2/2\sigma^2}}{\sqrt{2\pi\sigma^2}} \\ \eta(u) &= \lambda(\cos au - 1) - \frac{\sigma^2 u^2}{2} \end{aligned}$$

Here the law is again a mixture of normal laws $\mathfrak{N}(na, \sigma^2)$, $n = 0, \pm 1, \dots$ which however are now centered around integer multiples of a . The generator finally is

$$[Av](x) = \frac{\sigma^2}{2} \partial_x^2 v(x) + \lambda \frac{v(x+a) - 2v(x) + v(x-a)}{2}$$

because the integral jump term reduces itself to a finite difference term.

2.5 The relativistic *qm* laws $\mathfrak{R}(\lambda)$

The family of the relativistic *qm* (quantum mechanics) laws on the other hand is a particular case of the well known (centered and symmetric) Generalized–Hyperbolic family (see for example [17] and references quoted therein): in fact we have $\mathfrak{R}(\lambda) = \mathfrak{GH}(-\frac{1}{2}, 1, \lambda)$, as can be seen by direct inspection of their *pdf*'s and *chf*'s. Remark as a consequence that these are not simple Hyperbolic laws that constitute the different particular sub-family $\mathfrak{GH}(1, 1, \lambda)$. The name follows from the fact that – for a suitable identification of the parameters λ and a by means of the particle mass m , the velocity of light c and the Planck constant \hbar – its pseudo-differential generator

$$A = \eta(\partial_x) = mc^2 - \sqrt{m^2 c^4 - c^2 \hbar^2 \partial_x^2}$$

coincides with the Hamiltonian operator of the simplest form of a free relativistic Schrödinger equation [1, 12], and hence its corresponding *L-S* equation exactly coincides with this free relativistic Schrödinger equation (see Appendix D.5). $\mathfrak{R}(\lambda)$ is

closed under convolution, as can be seen from the form of the *chf*'s, but the laws are not stable for the same reasons as the Variance–Gamma: the parameter λ is not a scale parameter. The *pdf*'s and *chf*'s are explicitly known (see Table 3), and all their moments exist: the odd moments (in particular the expectation) vanish by symmetry, while the even moments are always finite and its variance is λ . Since the Lévy measure is explicitly known (see Table 3) the Lévy dimensionless generator also takes the form

$$[Av](x) = \lambda \int_{y \neq 0} [v(x+y) - v(x)] \frac{K_1(|y|)}{\pi|y|} dy.$$

where K_1 is a modified Bessel function.

3 The Lévy–Schrödinger equation

To keep the notations as simple as possible also in this chapter the laws and the time coordinate will again be supposed dimensionless. It has been shown in [1] that the evolution equation (9) of a centered, symmetric Lévy process can be formally turned into a *L-S* equation: in fact the pseudo-differential generator $\eta(\partial)$ of our processes is a self-adjoint operator in L^2 and hence can correctly play the role of a hamiltonian. We summarize in the following the formal steps leading to the *L-S* equation (for further details see [1]); this will also establish the notation for the subsequent sections.

Take as background noise a centered, symmetric, *id* law with f , $\varphi = e^\eta$, $\mathcal{L} = (0, \beta, \ell)$ and a symmetric ℓ so that (3) holds

$$\eta(u) = -\frac{\beta^2}{2} u^2 + \int_{y \neq 0} (\cos uy - 1) \ell(y) dy;$$

remember that since η is real and symmetric, φ too will be real, symmetric and non negative ($\varphi \geq 0$). Define then the transition *chf* $\chi(u, t) = \varphi^t(u)$ and the reduced transition *pdf*

$$q(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \varphi^t(u) e^{-iux} du$$

of the corresponding Lévy process, and take an initial law f_0 , $\varphi_0 = e^{\eta_0}$: the *chf* and the *pdf* of the process will be

$$\begin{aligned} \phi(u, t) &= \chi(u, t) \varphi_0(u) \\ p(x, t) &= [q(t) * f_0](x) = \int_{-\infty}^{+\infty} q(x-y, t) f_0(y) dy \\ p(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(u, t) e^{-iux} du. \end{aligned} \tag{12}$$

There are hence two ways to calculate $p(x, t)$: either as $p = q * f_0$, or by inverting the *chf* $\phi = \chi\varphi_0$. As a matter of fact these two ways give the same result, but – depending on the specific problem – one can be easier to calculate than the other. The *pdf* $p(x, t)$ of the previous step must also be a solution of the (dimensionless) evolution equation

$$\partial_t p(x, t) = \frac{\beta^2}{2} \partial_x^2 p(x, t) + \int_{y \neq 0} [p(x + y, t) - p(x, t)] \ell(y) dy \quad (13)$$

and in principle we could find p also by directly solving this equation.

We pass then to the *L-S* propagators by means of the formal substitution $t \rightarrow it$:

$$\gamma(u, t) = \chi(u, it) = \varphi^{it}(u) = e^{it\eta(u)}, \quad g(x, t) = q(x, it)$$

so that g and γ will still verify the same reciprocity relations (7) of q and χ

$$g(x, t) = q(x, it) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \chi(u, it) e^{-iux} du = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \gamma(u, t) e^{-iux} du.$$

Remark that if the law of the background noise is centered, symmetric and *id* then η is real, symmetric and positive and hence we always have $|\gamma| = 1$. This implies first that γ is not normalizable in L^2 , and hence that also g is not normalizable in L^2 . This is not surprising since, as it is well known, the propagators are not supposed to be normalizable *wf*'s. On the other hand, as we will see later, this also entails that an initial normalized *wf* will stay normalized all along its evolution. We choose now an initial *L-S wf*: to compare the evolutions of the *wf*'s with that of the process *pdf*'s, we will start – whenever we can – with a law f_0 , $\varphi_0 = e^{\eta_0}$ and with a *wf* ψ_0 such that $|\psi_0|^2 = f_0$, namely

$$\psi_0(x) = \sqrt{f_0(x)} e^{iS_0(x)}$$

where S_0 is an arbitrary, dimensionless, real function. In this way we are also sure that $\psi_0 \in L^2(\mathbb{R})$, and that $\|\psi_0\|^2 = 1$. As a matter of fact we could also characterize our initial state through the *wf FT* $\hat{\psi}_0(u)$ which exists because $\psi_0 \in L^2(\mathbb{R})$. It is possible to show that φ_0 and $\hat{\psi}_0$ must satisfy the following relation

$$\overline{\varphi}_0 = \hat{\psi}_0 * \hat{\psi}_0$$

which is simply the dual of $|\psi_0|^2 = f_0$. The initial *wf* can be simplified by choosing f_0 and φ_0 centered and symmetric, with $S_0 = 0$. In this way we will have real φ_0 and ψ_0 , with

$$\psi_0(x) = \sqrt{f_0(x)}, \quad (14)$$

so that the following relation will always be satisfied

$$\varphi_0 = \hat{\psi}_0 * \hat{\psi}_0. \quad (15)$$

Now the L - S wf 's will obey the following evolution scheme

$$\begin{aligned}\hat{\psi}(u, t) &= \gamma(u, t)\hat{\psi}_0(u) \\ \psi(x, t) &= [g(t) * \psi_0](x) = \int_{-\infty}^{+\infty} g(x - y, t)\psi_0(y) dy \\ \psi(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{\psi}(u, t)e^{iux} du\end{aligned}\tag{16}$$

Here we can see the relevance of having $|\gamma|^2 = 1$ (namely of having a centered, symmetric background Lévy noise, and hence a self-adjoint generator): we have indeed that $|\hat{\psi}(t)|^2 = |\gamma|^2|\psi_0|^2 = |\psi_0|^2$, so that if $\|\psi_0\|^2 = 1$ then also $\|\hat{\psi}(t)\|^2 = 1$, and as a consequence (by Parseval and Plancherel theorems) $\|\psi(t)\|^2 = 1$ at every t . In other words we can say that the non normalizability of the propagator is the counterpart of the unitarity of the L - S evolution. Finally the wf 's $\psi(x, t)$ introduced in the previous steps must satisfy the free L - S equation

$$i\partial_t\psi(x, t) = -\frac{\beta^2}{2}\partial_x^2\psi(x, t) - \int_{y \neq 0} [\psi(x + y, t) - \psi(x, t)] \ell(y) dy.\tag{17}$$

4 Processes and wave packets

We will give now several examples of L - S wf 's compared with the corresponding purely Lévy evolutions. We classify these examples first by choosing the laws of the background noises: this will be done by picking up the *id* laws that allow a reasonable knowledge of both the transition *pdf* of the Lévy process, and the L - S propagator. Besides the usual Wiener case (that will be considered just to show the way) this will indeed allow us to calculate the evolutions by means of integrations, without being obliged to solve pseudo-differential equations. The equation will be used instead – when it is possible – as a check on the solutions found from transition *pdf*'s and propagators. We will compare then the typical evolutions of the Lévy process *pdf*'s, and of the wf 's solutions of a free L - S equation: for details, notations and formulas about both the initial laws and wf 's, and the transition *pdf*'s and propagators we will make due references to Appendix C and to Appendix D. Remark also that in the following we will reintroduce the dimensional parameters a, b and τ .

4.1 Gauss

Take a Wiener process with transition law (48): for a normal initial law (35) \mathfrak{N}_b we have

$$\phi(u, t) = \chi(u, t)\varphi_0(u) = e^{-(2Dt+b^2)u^2/2}$$

so that the evolution is always Gaussian $\mathfrak{N}(2Dt+b^2)$: it starts with a non degenerate normal distribution of variance b^2 and then widens as the usual diffusions do with variance $2Dt+b^2$. The L - S evolution of the wf 's on the other hand is here the usual

quantum mechanical one: take first as initial *wf* the Gaussian (36): we then have as wave packets

$$\begin{aligned}\hat{\psi}(u, t) &= \gamma(u, t)\hat{\psi}_0(u) = \sqrt[4]{\frac{2b^2}{\pi}} e^{-(b^2+iDt)u^2} \\ \psi(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{\psi}(u, t) e^{iux} du = \sqrt[4]{\frac{b^2}{2\pi}} \frac{e^{-x^2/4(b^2+iDt)}}{\sqrt{b^2+iDt}}\end{aligned}$$

It is well known that in this case $|\psi(x, t)|^2$ has a widening, Gaussian shape all along its evolution. We neglect to display pictures of these well known evolutions.

4.2 Cauchy

The Cauchy process is one of the most studied non Gaussian, Lévy processes [6], first because it is stable, and then because the calculations are relatively accessible. For example, if the initial law is a Cauchy \mathfrak{C}_b with $\chi(u, t) = e^{-ct|u|}$, from (50) and (37) we immediately have for the transition *chf*

$$\phi(u, t) = e^{-(b+ct)|u|}$$

namely the process law remains a Cauchy \mathfrak{C}_{b+ct} at every t with a typical broadening for $t \rightarrow +\infty$

$$p(x, t) = \frac{1}{\pi} \frac{b+ct}{(b+ct)^2 + x^2}. \quad (18)$$

Of course this behavior (which is in common with the Gaussian Wiener process) comes out from the fact that the Cauchy laws are stable, and we neglect to display the corresponding figure. Even when the initial *pdf* is a $\mathfrak{T}_b(3)$ with $\varphi_0(u) = (1 + b|u|)e^{-b|u|}$ calculations are easy: now the transition law is again \mathfrak{C}_{ct} , and the one-time process law $\mathfrak{C}_{ct} * \mathfrak{T}_b(3)$ will have as *chf*

$$\phi(u, t) = \chi(u, t)\varphi_0(u) = (1 + b|u|)e^{-(b+ct)|u|}$$

while the *pdf* is recovered by *chf* inversion:

$$p(x, t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(u, t) e^{-iux} du = \frac{(b+ct)^2(2b+ct) + vtx^2}{\pi [(b+ct)^2 + x^2]^2}. \quad (19)$$

It would be easy to check that this is again a normalized, uni-modal, bell-shaped, broadening *pdf* (see Fig. 3), with neither an expectation nor a finite variance for $t > 0$. We would find in particular that the process law is the mixture

$$\mathfrak{C}_{ct} * \mathfrak{T}_b(3) = \frac{1}{2} \frac{ct}{b+ct} \tilde{\mathfrak{B}}_{b+ct}^{1/2} \left(\frac{3}{2}, \frac{1}{2} \right) + \frac{1}{2} \frac{2b+ct}{b+ct} \tilde{\mathfrak{B}}_{b+ct}^{1/2} \left(\frac{1}{2}, \frac{3}{2} \right) \quad (20)$$

of the laws $\tilde{\mathfrak{B}}_a^{1/2}(\alpha, \beta)$ of the square root of second kind Beta *rv*'s (see Appendix E for details). Remark that in particular $\tilde{\mathfrak{B}}_{b+ct}^{1/2}(1/2, 3/2) = \mathfrak{T}_{b+ct}(3)$. For this example

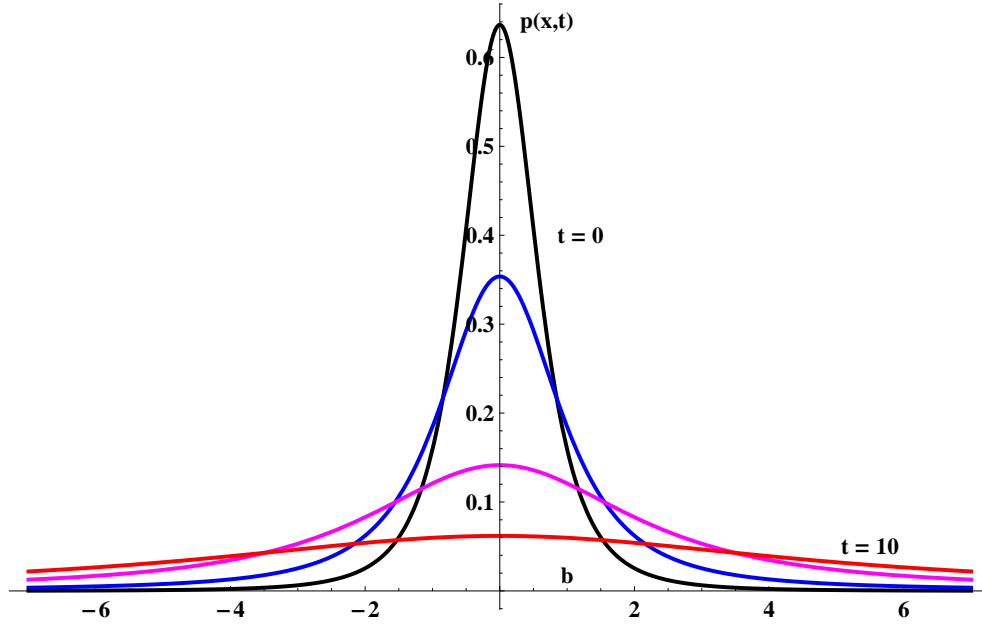


Figure 3: The *pdf* (19) for a Cauchy process with a Student $\mathfrak{T}_b(3)$ initial distribution.

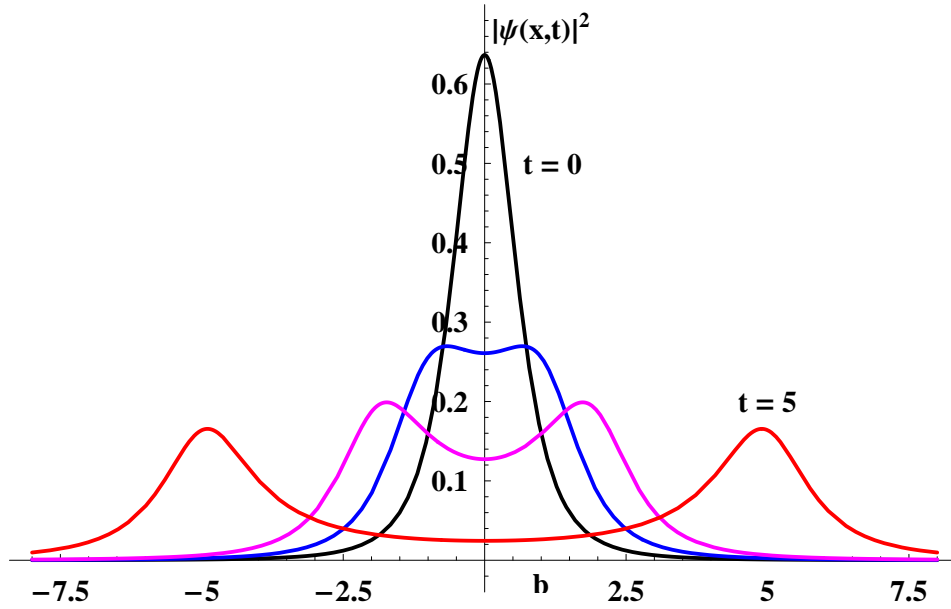


Figure 4: The square modulus of the Cauchy–Schrödinger *wf* (21) for a Student $\mathfrak{T}_b(3)$ initial distribution.

we can also show by direct calculation that the *pdf*'s (18) and (19) are both solutions of the pseudo–differential Cauchy equation (51).

The Cauchy–Schrödinger evolutions, on the other hand, show a more interesting structure. The simplest case is found when we take as $|\psi_0|^2$ the Student $\mathfrak{T}_b(3)$ case (40): from (52) indeed we have

$$\hat{\psi}(u, t) = \gamma(u, t)\hat{\psi}_0(u) = \sqrt{b} e^{-(b+ict)|u|}$$

and hence

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \hat{\psi}(u, t) e^{iux} du = \sqrt{\frac{2b}{\pi}} \frac{b + ict}{(b + ict)^2 + x^2} \quad (21)$$

This *wf* (see Figure 4) is correctly normalized in L^2 but shows a new feature: *bi-modality*. In fact $|\psi|^2$ has now two well defined maxima smoothly drifting away from the center as $t \rightarrow +\infty$. It is also possible to show – as an example – that our *wf* is a solution of the Cauchy–Schrödinger equation (53). For the right–hand side of this equation we indeed have from the principal value integral

$$- \int_{y \neq 0} [\psi(x + y, t) - \psi(x, t)] \frac{c}{\pi y^2} dy = c \sqrt{\frac{2b}{\pi}} \frac{(b + ict)^2 - x^2}{[(b + ict)^2 + x^2]^2}$$

which is easily seen to coincide with $i\partial_t \psi(x, t)$. As a consequence the *wf* (21) correctly satisfies the pseudo–differential Cauchy–Schrödinger equation (53). A similar result is found in the case of a Cauchy \mathfrak{C}_b initial *wf* (38): from the propagator (52) we have

$$\hat{\psi}(u, t) = \frac{\sqrt{2b}}{\pi} K_0(b|u|) e^{-ict|u|}$$

and hence by inverting the *FT*:

$$\begin{aligned} \psi(x, t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\sqrt{2b}}{\pi} K_0(b|u|) e^{-ict|u|} e^{iux} du \\ &= \frac{1}{\pi \sqrt{b\pi}} \left[A\left(\frac{x + ct}{b}\right) + \overline{A\left(\frac{x - ct}{b}\right)} \right] \end{aligned} \quad (22)$$

where we defined

$$A(z) = \frac{\frac{\pi}{2} - i \operatorname{arcsinh} z}{\sqrt{1 + z^2}}$$

and we used the following two results

$$\int_0^{+\infty} \cos(xz) K_0(z) dz = \frac{\pi}{2} \frac{1}{\sqrt{1 + x^2}}, \quad \int_0^{+\infty} \sin(xz) K_0(z) dz = \frac{\operatorname{arcsinh} x}{\sqrt{1 + x^2}}.$$

The *wf* (22) is normalized in L^2 and shows (see Figure 5) a behavior similar to that of (21): its *pdf* $|\psi|^2$ starts as a Cauchy \mathfrak{C}_b distribution and then widens with two well defined maxima drifting away from the center. Here too, hence, we have bi-modality: remark the difference with the Cauchy process *pdf*'s \mathfrak{C}_{b+ct} and $\mathfrak{C}_{ct} * \mathfrak{T}_b(3)$ which instead broaden by remaining strictly unimodal.

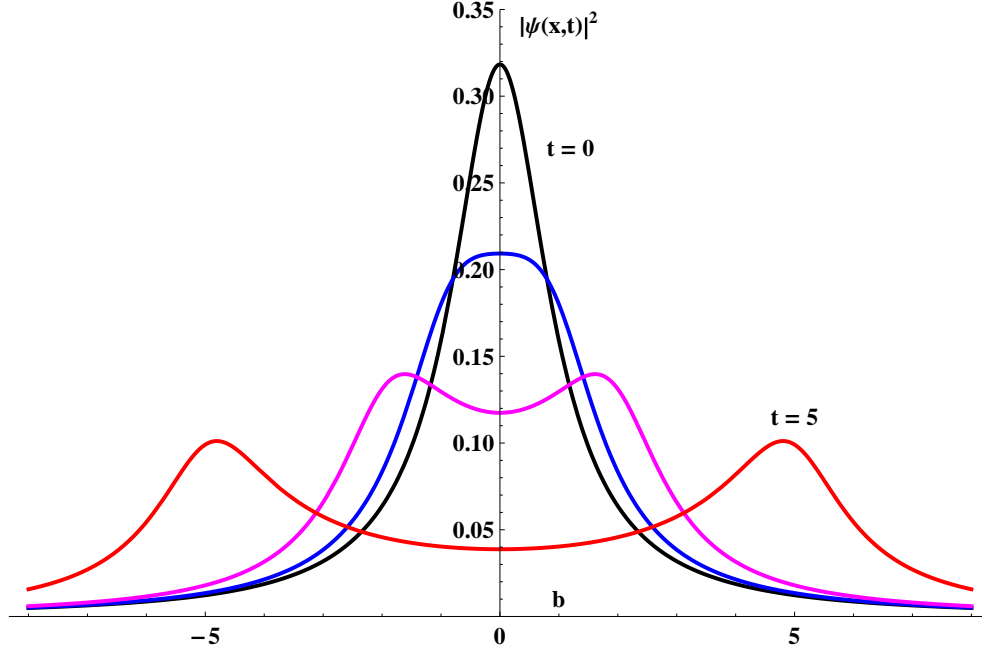


Figure 5: The square modulus of the Cauchy–Schrödinger *wf* (22) for a Cauchy \mathfrak{C}_b initial distribution.

4.3 Laplace

This bi-modality of the wave packets, or at least its breaking in two symmetric structures drifting away from the center can also be found in other examples. Take first the Variance–Gamma process of Appendix D.3. At variance with the Cauchy process, this is an example of a non stable, *sd* process and hence has a certain interest as a non typical case. We will refer to the Appendix C.4 for a discussion of possible initial states. At present we will limit our discussion to initial states of the same Variance–Gamma family of the background noise, and we will also always choose coincident scale parameters $a = b$ for the background noise and the initial states.

For a Variance–Gamma process with transition law (54) and initial *pdf* (41) we immediately have

$$\phi(u, t) = \chi(u, t)\varphi_0(u) = \left(\frac{1}{1 + b^2 u^2}\right)^{\nu + \omega t}$$

and hence the process law simply is $\mathfrak{VG}_b(\nu + \omega t)$ with *pdf*

$$p(x, t) = \frac{2}{2^\nu \Gamma(\nu) \sqrt{2\pi} b} \left(\frac{|x|}{b}\right)^{\nu + \omega t - \frac{1}{2}} K_{\nu + \omega t - \frac{1}{2}} \left(\frac{|x|}{b}\right) \quad (23)$$

namely always a Variance–Gamma but with a growing parameter $\nu + \omega t$. On the one hand this explains why it would be delusory to think of simplifying the example

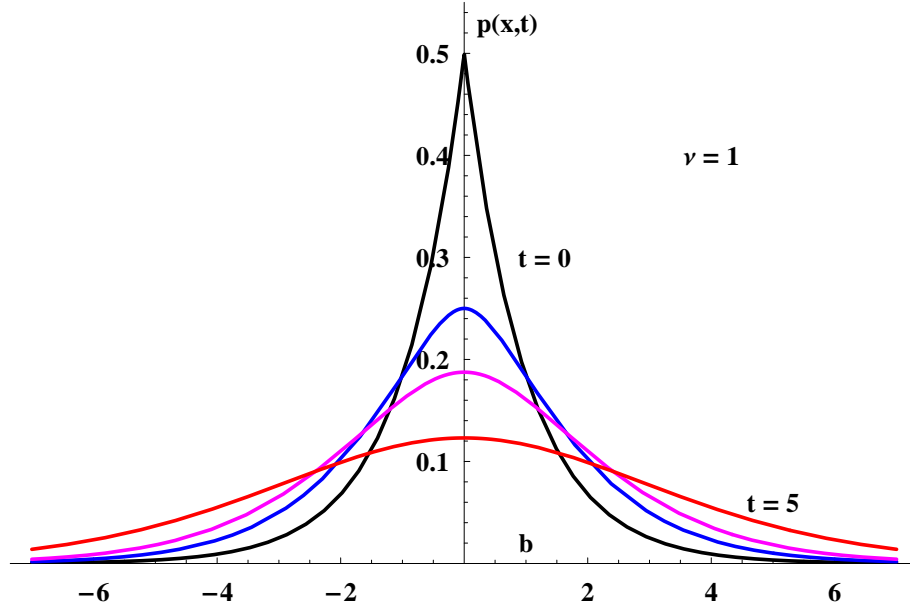


Figure 6: The pdf (23) for a Variance–Gamma process with Laplace $\mathfrak{VG}_b(1) = \mathfrak{L}_b$ initial distribution.

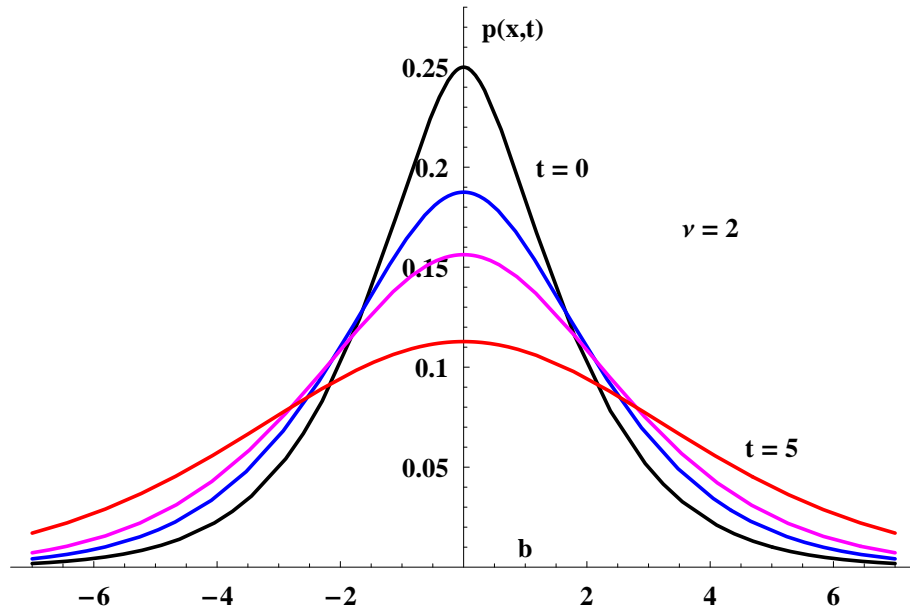


Figure 7: The pdf (23) for a Variance–Gamma process with Variance–Gamma $\mathfrak{VG}_b(2)$ initial distribution.

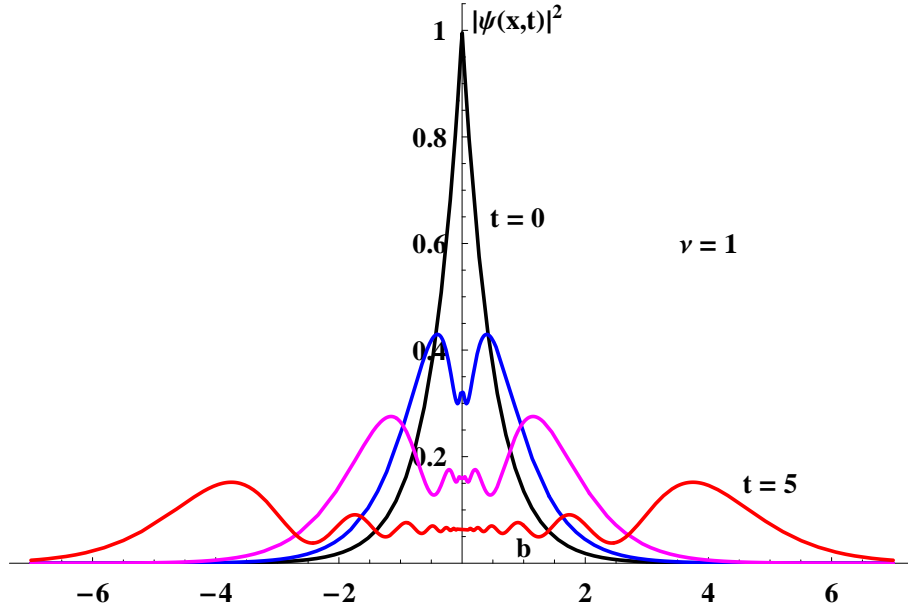


Figure 8: The square modulus of the Variance–Gamma–Schrödinger wf (24) with a Laplace $\mathfrak{VG}_b(1) = \mathfrak{L}_b$ initial wf .

by starting, for instance, with a Laplace $\mathfrak{L}_b = \mathfrak{VG}_b(1)$ initial law: in fact at every time $t > 0$ the process law would in any case no longer be a Laplace law, but a more general Variance–Gamma with $\nu + \omega t \neq 1$. On the other hand this apparently explains why at every t the pdf will appear as a broadening, uni-modal distribution as shown in the Figures 6 and 7 respectively for $\nu = 1$ and $\nu = 2$.

For a L - S evolution, on the other hand, we have from (55) and (43)

$$\hat{\psi}(u, t) = \sqrt{\frac{b}{\sqrt{\pi}} \frac{\Gamma(2\nu)}{\Gamma(2\nu - \frac{1}{2})}} \left(\frac{1}{1 + b^2 u^2} \right)^{\nu + i\omega t}$$

so that the inverse FT will be

$$\begin{aligned} \psi(x, t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{\psi}(u, t) e^{iux} du \\ &= \sqrt{\frac{b}{\sqrt{\pi}} \frac{\Gamma(2\nu)}{\Gamma(2\nu - \frac{1}{2})}} \frac{2}{2^{\nu + i\omega t} \Gamma(\nu + i\omega t) \sqrt{2\pi}} \\ &\quad \frac{1}{b} \left(\frac{|x|}{b} \right)^{\nu + i\omega + 1/2} K_{\nu + i\omega + 1/2} \left(\frac{|x|}{b} \right) \end{aligned} \quad (24)$$

Numerical calculations and plotting then show that the wf (24) always is normalized, and that $|\psi|^2$ has two maxima symmetrically drifting away from the center (see Figure 8). The behavior in $x = 0$ is rapidly oscillating, but with infinitesimal

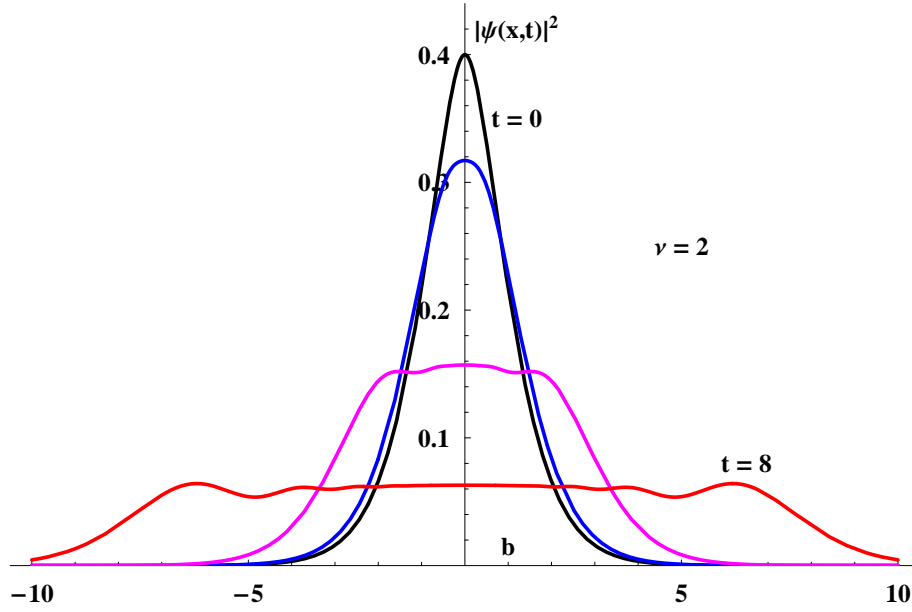


Figure 9: The square modulus of the Variance–Gamma–Schrödinger wf (24) with a Variance–Gamma $\mathfrak{VG}_b(2)$ initial wf .

amplitude as we approach $x = 0$: in fact the singular behavior of the Bessel function is here competing with an infinitesimal $|x|^\nu$ factor. The distribution shows also a slowly decreasing, flat plateau (with micro-oscillations) in the central region, while the diverging maxima can be rather dull as in the Figure 9.

4.4 Poisson

The following examples come from two ac , but not sd background noises: the compound Wiener–Poisson processes introduced in the Appendix D.4. First take the process with the transition law $\mathfrak{N}(2Dt) * \mathfrak{P}(\omega t, \mathfrak{N}_a)$ in (56): with a normal initial law (35) the marginal law of the process becomes $\mathfrak{N}(2Dt + b^2) * \mathfrak{P}(\omega t, \mathfrak{N}_a)$ namely

$$p(x, t) = e^{-\omega t} \sum_{k=0}^{\infty} \frac{(\omega t)^k}{k!} \frac{e^{-x^2/2(ka^2+2Dt+b^2)}}{\sqrt{2\pi(ka^2+2Dt+b^2)}} \quad (25)$$

which apparently is a Poisson mixture of centered, normal pdf 's of different variances, and hence has the usual bell-like, uni-modal, diffusing shape that we will not bother to show. For the other transition law $\mathfrak{N}(2Dt) * \mathfrak{P}(\omega t, \mathfrak{D}_a)$ in (58) with the same normal initial distribution the marginal law instead is $\mathfrak{N}(2Dt + b^2) * \mathfrak{P}(\omega t, \mathfrak{D}_a)$ namely

$$p(x, t) = e^{-\omega t} \sum_{k=0}^{\infty} \frac{(\omega t)^k}{k!} \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} \frac{e^{-[x-(k-2j)a]^2/2(2Dt+b^2)}}{\sqrt{2\pi(2Dt+b^2)}}. \quad (26)$$

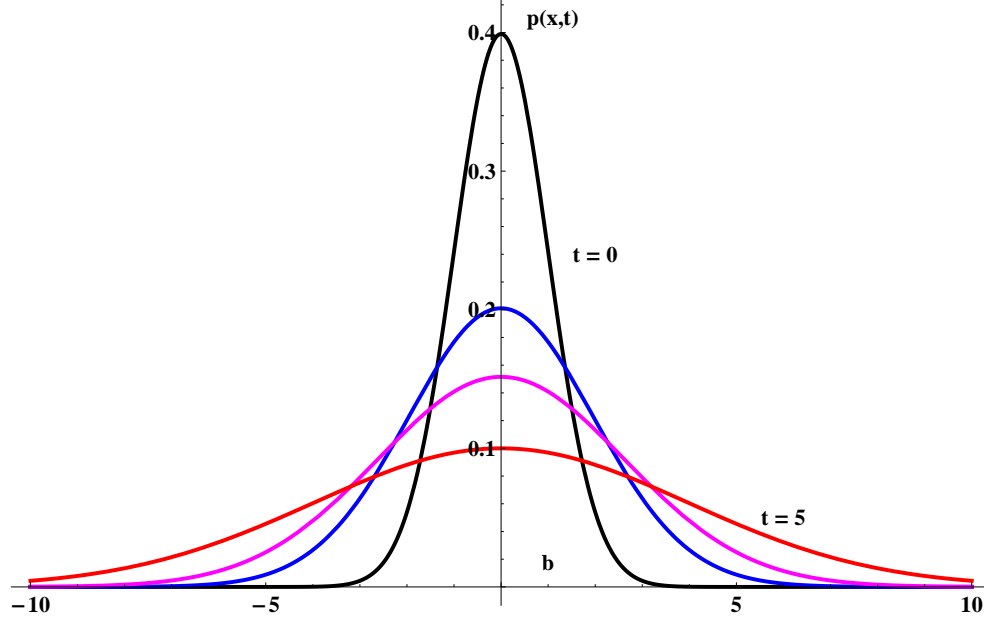


Figure 10: The pdf (26) of a Normal–Poisson process $\mathfrak{N}(2Dt) * \mathfrak{P}(\omega t, \mathfrak{D}_a)$ with a Gaussian initial law.

In other words we always have generalized Poisson mixtures, but of non centered normal pdf 's. Even in this case, however, the shape of the overall pdf will be that of a bell-like, uni-modal, diffusing curve (see Figure 10).

For the L - S equation on the other hand consider first the propagator $\mathfrak{N}(2iDt) * \mathfrak{P}(i\omega t, \mathfrak{N}_a)$ in (57) applied to an initial Gaussian wf (36); we then have

$$\hat{\psi}(u, t) = e^{i\omega t(e^{-a^2 u^2/2} - 1)} \sqrt[4]{\frac{2b^2}{\pi}} e^{-(b^2 + iDt)u^2}$$

and, by inverting the FT and taking into account the properties of the Gaussian integrals, the wf will be

$$\psi(x, t) = e^{i\omega t} \sum_{k=0}^{\infty} \frac{(i\omega t)^k}{k!} \sqrt[4]{8\pi b^2} \frac{e^{-x^2/2(ka^2 + 2b^2 + 2iDt)}}{\sqrt{2\pi(ka^2 + 2b^2 + 2iDt)}} \quad (27)$$

namely a time-dependent, complex, Poisson superposition of centered Gaussian wf 's. The same is true for the second example with propagator $\mathfrak{N}(2iDt) * \mathfrak{P}(i\omega t, \mathfrak{D}_a)$ in (59) with an initial Gaussian wf (36): the wf FT in fact now is

$$\hat{\psi}(u, t) = e^{i\omega t(\cos au - 1)} \sqrt[4]{\frac{2b^2}{\pi}} e^{-(b^2 + iDt)u^2}$$

so that the wf itself will be

$$\psi(x, t) = e^{i\omega t} \sum_{k=0}^{\infty} \frac{(i\omega t)^k}{k!} \frac{\sqrt[4]{8\pi b^2}}{2^k} \sum_{j=0}^k \binom{k}{j} \frac{e^{-[x - (k-2j)a]^2/4(b^2 + iDt)}}{\sqrt{4\pi(b^2 + iDt)}}. \quad (28)$$

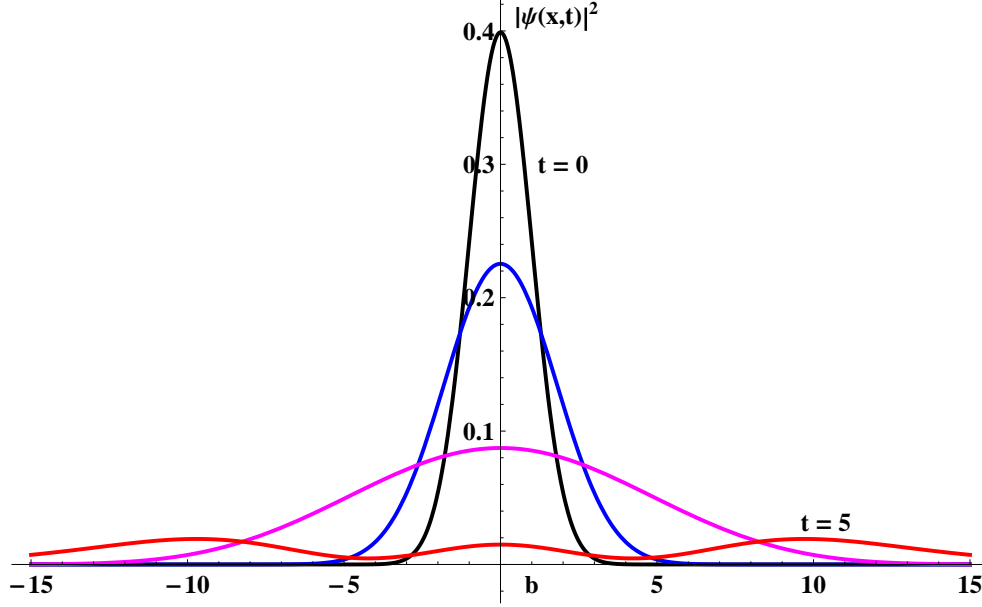


Figure 11: The square modulus of the Normal–Poisson Schrödinger *wf* (28) with a Gaussian initial *wf*.

In conclusion, while the plots of $p(x, t)$ in (25) and (26) simply display the too familiar story of a diffusing bell-shaped curve, and the same would be true for $|\psi(x, t)|^2$ in (27), for $|\psi(x, t)|^2$ in (28) we instead have again a separation of the wave packet in two symmetrical sub-packets drifting away from the center (see Figure 11).

4.5 Relativistic qm

In a way similar to that of the Variance–Gamma, for a Relativistic qm Lévy process with transition law (60) and initial distribution (46), but with $a = b$, we immediately have

$$\phi(u, t) = \chi(u, t)\varphi_0(u) = e^{(\nu+\omega t)(1-\sqrt{1+a^2u^2})} \quad (29)$$

$$p(x, t) = \frac{(\nu + \omega t)e^{\nu+\omega t}}{\pi a} \frac{K_1\left(\sqrt{(\nu + \omega t)^2 + x^2/a^2}\right)}{\sqrt{(\nu + \omega t)^2 + x^2/a^2}} \quad (30)$$

and hence the process law simply is $\mathfrak{R}(\nu + \omega t)$, namely it will stay always in the same Relativistic qm family but with a time dependent parameter. The *pdf* $p(x, t)$ is shown in the Figure 12 and has the usual bell-like, uni-modal, diffusing form. For the corresponding L - S evolution on the other hand we have from (47) and (61)

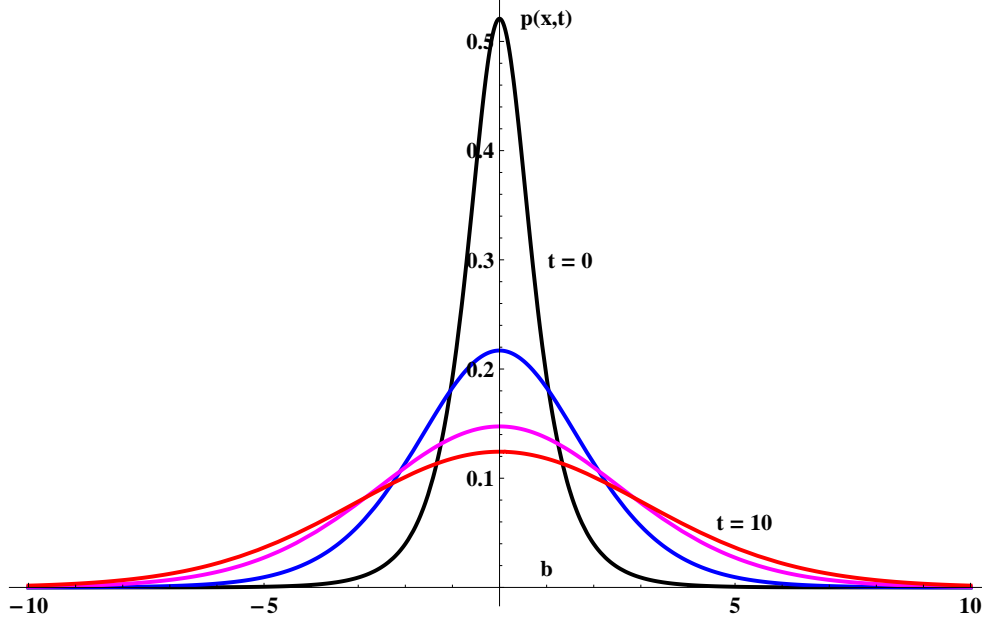


Figure 12: The *pdf* (30) of a Lévy process with a Relativistic *qm* background noise and an initial law of the same family.

that the normalized *wf*'s are

$$\hat{\psi}(u, t) = \gamma(u, t) \hat{\psi}_0(u) = \sqrt{\frac{a}{2e^{2\nu} K_1(2\nu)}} e^{(\nu + i\omega t)(1 - \sqrt{1 + a^2 u^2})} \quad (31)$$

$$\psi(x, t) = \frac{(\nu + i\omega t)e^{i\omega t}}{\sqrt{a\pi K_1(2\nu)}} \frac{K_1\left(\sqrt{(\nu + i\omega t)^2 + x^2/a^2}\right)}{\sqrt{(\nu + i\omega t)^2 + x^2/a^2}} \quad (32)$$

We show in the Figure 13 how this $|\psi(x, y)|^2$ behaves, and in particular, at variance with the previous Lévy *pdf* (30), we find here again that the *wf* shows two symmetric maxima drifting away from the center of the distribution: the bi-modality that we have already pointed out in all our other *L-S* examples.

5 Conclusions

We presented in the previous sections several examples of free wave packets that are solutions of the *L-S* equation without potentials (17). We started by generalizing the relation between Brownian motion and Schrödinger equation, and by associating the kinetic energy of a physical system to the generator of a symmetric Lévy process, namely to a pseudo-differential operator whose symbol is the *lch* η of an *id* law. This amounts to suppose, then, that the *L-S* equation is based on an underlying Lévy process that can have both Gaussian (continuous) and non Gaussian (jumping)

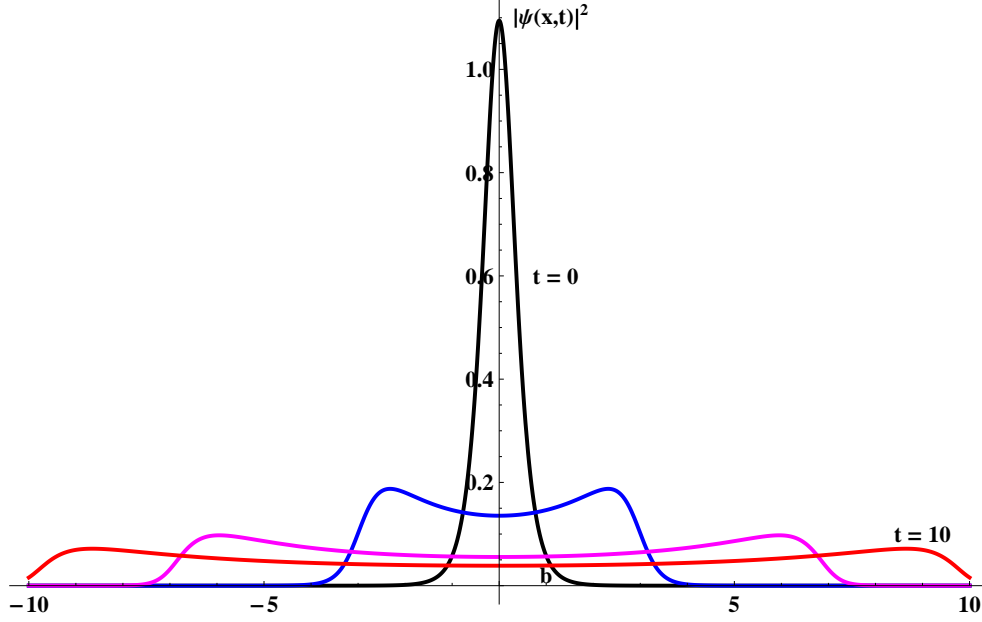


Figure 13: The square modulus of the Relativistic qm wf (32) with an initial wf of the same family.

components. The use of all the id , even non stable, processes on the other hand is important and physically meaningful because there are significant cases that are in the domain of our L - S picture, without being in that of the stable (fractional) Schrödinger equation. In particular, as discussed in [1, 6], the simplest form of a relativistic, free Schrödinger equation can be associated with a particular type of sd , non stable process acting as background noise. Moreover in many instances of the Lévy–Schrödinger equation the new energy–momentum relations can be seen as corrections to the classical relations for small values of certain parameters [1]. It must also be remembered that – at variance with the stable, fractional case – our model is not tied to the use of processes with infinite variance: the variances can be chosen to be finite even in a purely non Gaussian model – as in the case of the relativistic, free Schrödinger equation – and can then be used as a legitimate measure of the dispersion. Finally let us recall that a typical non stable, Student Lévy noise seems to be suitable for applications in the models of halo formation in intense beam of charged particles in accelerators [8, 17, 22].

It was then important to explore the general behavior of the diffusing L - S wf 's: we systematically approached this problem by defining in Section 3 a procedure allowing us to explore several combinations of initial wf 's (Appendix C) and background Lévy noises (Appendix D), and by comparing Lévy processes and free L - S wave packets. We have then remarked that virtually in all our examples of Section 4 we witnessed a similar qualitative behavior: first of all the L - S wave packets diffuse, in the sense that they broaden in a very regular way. As it is known the variance

of a Lévy process – when it exists – grows linearly with the time, exactly as in the usual diffusions. Of course stable, non Gaussian noises are excluded, since for them there is no variance, and we have instead an anomalous sub- and super-diffusive behavior. The corresponding L - S wave packets show a similar qualitative behavior also if it is not always easy to calculate their variances.

A second, more surprising feature however is represented by the bi-modality of the L - S wf 's. In fact we found that in virtually all our examples the wave packet splits in two sub-packets symmetrically and smoothly drifting away from the center: a behavior which is present neither in the free Lévy processes, nor in the (Gaussian) free Schrödinger wf 's. It is interesting to remark, then, that the unique instance with a similar bi-modal behavior has been found earlier [10] deals with *confined* Lévy flights. In our opinion the bi-modality found in our examples could then be connected to the combined effect of Nelson dynamics, and Lévy jumps in the background noise, and it would be interesting to explore if this behavior shows up again in form of rings and shells respectively for the two- and three-dimensional L - S equation. This bi-modality, on the other hand, is in sheer contrast with the uni-modality of both the Lévy processes and the (Gaussian) Schrödinger wf 's.

It would be important now to explicitly give in full detail the formal association between L - S wf 's and the underlying Lévy processes, namely a true generalized stochastic mechanics. In particular we would show that to every wf solution of the L - S equation we can associate a well defined Lévy process: the techniques of the stochastic calculus applied to Lévy processes are today in full development [11, 12, 23], and at our knowledge there is no apparent, fundamental impediment along this road. Finally it would be relevant to explore this Lévy–Nelson stochastic mechanics by adding suitable potentials to our L - S equation, and by studying the corresponding possible stationary and coherent states: all that too will be the subject of future papers.

A Types of laws

As stated in the Section 1 we deal in this paper with centered laws of rv 's X . Even when the expectation does not exist we can always speak of centering around the *median*. On the other hand to eliminate the centering it will be enough to take $X + b$ with $b \in \mathbb{R}$ instead of X , then to substitute $x - b$ to x in the f , and to add a factor e^{ibu} to the chf φ . For our purposes it will also be expedient to introduce a dimensional *scale parameter* $a > 0$ to take into account the physical dimensions of our rv 's: to fix the ideas in this paper a will be supposed to be a *length*. Take first a rv X with law \mathfrak{F} , pdf f and chf φ , and suppose that X is a dimensionless quantity; then the variables argument of f and φ , will be dimensionless. On the other hand $X_a = aX$ will be a length and will follow a law \mathfrak{F}_a with

$$f_a(x) dx = f\left(\frac{x}{a}\right) \frac{dx}{a}, \quad \varphi_a(u) = \varphi(au).$$

Here x and u are now dimensional variables (x is a length, while u is the reciprocal of a length), so that x/a and au will be dimensionless. Remark that within this notation we numerically have $\mathfrak{F} = \mathfrak{F}_1$, so that for instance $f_1(x) = f(x)$. This could be slightly misleading since the argument of f_1 is a length, while that of f is supposed to be dimensionless. To avoid any possible misunderstanding we will then reserve the symbols \mathfrak{F} , f and φ for the dimensionless laws, while \mathfrak{F}_1 , f_1 and φ_1 will be associated to the dimensional ones. For example if X follows the *standard*, dimensionless normal law \mathfrak{N} with

$$f(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}}, \quad \varphi(u) = e^{-u^2/2}$$

the dimensional rv 's $X_a = aX$ will follow the laws $\mathfrak{N}_a = \mathfrak{N}(a^2)$ with

$$f_a(x) = \frac{e^{-x^2/2a^2}}{a\sqrt{2\pi}}, \quad \varphi_a(u) = e^{-a^2u^2/2}.$$

Then f and f_1 will be coincident, but the dimensional meaning of their respective variables will be different. Remark finally that in general we will choose dimensionless laws that are not necessarily standard laws: of course (when the variances exist) we will have $\mathbf{V}[X_a] = a^2 \mathbf{V}[X]$, but $\mathbf{V}[X]$ is not always supposed to be equal to 1.

We could now think to \mathfrak{F}_a as the parametric family of the rescaled rv 's aX : these parametric families spanned just by one scale parameter a are here entire *types of laws*⁴: in fact, since here we only deal with centered laws (see Section 1), no centering parameter b is required, and our types are spanned by means of the scale parameter a only. In this paper we will also consider other parametric families of laws with some dimensionless parameter λ , which will not in general be coincident with the scale parameter a . We could then have two-parameters families $\mathfrak{F}_a(\lambda)$, and in general we are interested in finding which sets are closed under convolution (namely under addition of the corresponding independent rv 's). When a type of laws is closed under convolution (as in the normal case of the previous example) its laws are said to be *stable*: the convolution would produce another law of the same type, namely a law with only a different *scale* parameter (in our notation: same λ , but different a). If instead the convolution produces a law of the same family, but not of the same type (different λ), then the family is closed under convolution, but its laws are not stable: this is the case, among others, of the Variance–Gamma laws $\mathfrak{VG}_a(\lambda)$. Finally, when the result of a convolution is a law not belonging at all to the family, then $\mathfrak{F}_a(\lambda)$ is not even closed under convolution, as for the Student $\mathfrak{T}_a(\lambda)$ family.

⁴A *type of laws* (see [16] Section 14) is a family of laws that only differ among themselves by a centering and a rescaling: in other words, if $\varphi(u)$ is the *chf* of a law, all the laws of the same type have *chf*'s $e^{ibu}\varphi(au)$ with a centering parameter $b \in \mathbb{R}$, and a scaling parameter $a > 0$ (we exclude here the sign inversions). In terms of rv 's this means that the laws of X and $aX + b$ (for $a > 0$, and $b \in \mathbb{R}$) always are of the same type, and on the other hand that X and Y belong to the same type if and only if it is possible to find $a > 0$, and $b \in \mathbb{R}$ such that Y and $aX + b$ have the same law, namely $Y \stackrel{d}{=} aX + b$.

B Symmetric and *ac*, compound Poisson laws

Among the *id*, non *sd* laws the Poisson case stands as the most important example, but the simple Poisson law is neither symmetric, nor *ac*. We will then generalize it in order to avoid these shortcomings. A Poisson law $\mathfrak{P}(\lambda)$ is a non symmetric, non *sd*, non *ac*, *id* law without Gaussian component ($\beta = 0$). The probability is concentrated on the integer numbers with the usual Poisson distribution so that formally

$$f(x) = \sum_{k=0}^{\infty} e^{-\lambda} \frac{\lambda^k}{k!} \delta_k(x), \quad \varphi(u) = e^{\lambda(e^{iu}-1)}, \quad \ell(x) = \lambda \delta_1(x).$$

Both expectation and variance have value λ . Since $\mathfrak{P}(\lambda)$ is neither centered, nor symmetric the generator of the corresponding Lévy process will not be self-adjoint. It is well known, moreover, that the sample paths of the corresponding simple Poisson process are ascending staircase trajectories, with randomly located steps of unit height, λ representing the average number of jumps per unit time interval. As a consequence these processes are not *ac*. To move ahead we must then first symmetrize the Poisson law, and then make it *ac*.

Take a symmetric (we do not require it to be *ac* or *id*) law \mathfrak{H} with *chf* $\vartheta(u) = e^{\zeta(u)}$ and build the corresponding compound Poisson law $\mathfrak{P}(\lambda, \mathfrak{H})$ with *chf*

$$\varphi(u) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \vartheta^k(u) = e^{\lambda[\vartheta(u)-1]}, \quad \eta(u) = \lambda[\vartheta(u) - 1]$$

thus generalizing the simple Poisson case where $\vartheta(u) = e^{iu}$. When \mathfrak{H} is also *ac* with *pdf* $h(x)$ the law of $\mathfrak{P}(\lambda, \mathfrak{H})$ is

$$f(x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} h^{*k}(x), \quad h^{*k} = \begin{cases} \overbrace{h * \dots * h}^{k \text{ times}}, & k = 1, 2, \dots \\ \delta_0, & k = 0 \end{cases} \quad (33)$$

but we can immediately see that this is still not *ac* even if \mathfrak{H} has a density: in fact for $k = 0$ we always have a degenerate law δ_0 . The compound Poisson law $\mathfrak{P}(\lambda, \mathfrak{H})$ has neither a drift ($\alpha = 0$ because of the required symmetry) nor a Gaussian part ($\beta = 0$), and its Lévy *pdf* (that we will suppose for simplicity to show no singularities at $x = 0$) is $\ell(x) = \lambda h(x)$: namely we have $\mathcal{L} = (0, 0, \lambda h)$. The laws of the increments of the corresponding compound Poisson process $\mathfrak{P}(\omega t, \mathfrak{H})$ with $\omega = \lambda/\tau$ are then the time dependent mixtures

$$p(x, t) = e^{-\omega t} \sum_{k=0}^{\infty} \frac{(\omega t)^k}{k!} h^{*k}(x) \quad (34)$$

while its self-adjoint generator (no singularities are present at $x = 0$) is

$$[Av](x) = \lambda \int_{-\infty}^{+\infty} [v(x+y) - v(x)] h(y) dy.$$

Its sample trajectories are now up and down staircase functions, with steps at Poisson random times, and random jump heights distributed according to the symmetric law \mathfrak{H} . Since however for $k = 0$ the law is degenerate in $x = 0$, these sample trajectories stick at $x = 0$ for a finite time (with probability 1), and the marginal distribution of the process is not *ac*. In other Lévy processes instead (as the Wiener process for example) the trajectory starts at $x = 0$, but its random path immediately leaves this position.

To give a first example of these symmetric (but not *ac*) compound Poisson laws take $\mathfrak{H} = \mathfrak{N}_a$ so that $h^{*k} \sim \mathfrak{N}(ka^2)$ for $k = 0, 1, \dots$; we then have for $\mathfrak{P}(\lambda, \mathfrak{N}_a)$

$$f(x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{e^{-x^2/2ka^2}}{\sqrt{2\pi ka^2}}, \quad \eta(u) = \lambda \left(e^{-a^2 u^2/2} - 1 \right), \quad \ell(x) = \lambda \frac{e^{-x^2/2a^2}}{\sqrt{2\pi a^2}}.$$

The transition *pdf*'s of the corresponding compound Poisson process are then the time dependent mixtures of $\mathfrak{N}(ka^2)$ laws

$$p(x, t) = e^{-\omega t} \sum_{k=0}^{\infty} \frac{(\omega t)^k}{k!} \frac{e^{-x^2/2ka^2}}{\sqrt{2\pi ka^2}},$$

and the generator takes the form

$$[Av](x) = \lambda \int_{-\infty}^{+\infty} [v(x+y) - v(x)] \frac{e^{-y^2/2a^2}}{\sqrt{2\pi a^2}} dy.$$

As another example suppose instead that $\mathfrak{H} = \mathfrak{D}_a$ is a Bernoulli symmetric law, doubly degenerate around the positions $\pm a$, namely

$$h(x) = \frac{1}{2a} \left[\delta_1 \left(\frac{x}{a} \right) + \delta_{-1} \left(\frac{x}{a} \right) \right], \quad \vartheta(u) = \cos au,$$

and remark that now

$$h^{*k}(x) = \frac{1}{2^k a} \sum_{j=0}^k \binom{k}{j} \delta_{k-2j} \left(\frac{x}{a} \right).$$

As a consequence we will have for $\mathfrak{P}(\lambda, \mathfrak{D}_a)$:

$$\begin{aligned} f(x) &= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \frac{1}{2^k a} \sum_{j=0}^k \binom{k}{j} \delta_{k-2j} \left(\frac{x}{a} \right) \\ \eta(u) &= \lambda (\cos au - 1) \\ \ell(x) &= \frac{\lambda}{2a} \left[\delta_1 \left(\frac{x}{a} \right) + \delta_{-1} \left(\frac{x}{a} \right) \right] \end{aligned}$$

We then easily have for the transition law of the process

$$p(x, t) = e^{-\omega t} \sum_{k=0}^{\infty} \frac{(\omega t)^k}{k!} \frac{1}{2^k a} \sum_{j=0}^k \binom{k}{j} \delta_{k-2j} \left(\frac{x}{a} \right)$$

while the generator is

$$[Av](x) = \frac{\lambda}{2} [v(x+1) - 2v(x) + v(x-1)]$$

We will then further generalize our compound Poisson distributions in order to get *ac* laws and processes. Take a compound Poisson law $\mathfrak{P}(\lambda, \mathfrak{H})$, and another independent, symmetric, *ac*, *id* law \mathfrak{H}_0 with *pdf* $h_0(x)$, *chf* $\vartheta_0(u) = e^{\zeta_0(u)}$ and Lévy triplet $\mathcal{L}_0 = (0, \beta_0, \ell_0)$. Consider then the law $\mathfrak{H}_0 * \mathfrak{P}(\lambda, \mathfrak{H})$ obtained by addition (convolution) so that

$$\varphi(u) = \vartheta_0(u) e^{\lambda(\vartheta(u)-1)}, \quad \eta(u) = \zeta_0(u) + \lambda(\vartheta(u) - 1)$$

while the *pdf* is

$$f(x) = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} (h_0 * h^{*k})(x), \quad h_0 * h^{*k} = \begin{cases} h_0 * \overbrace{h * \dots * h}^{k \text{ times}}, & k = 1, 2, \dots \\ h_0, & k = 0 \end{cases}$$

This is now a mixture of *ac* laws. The law $\mathfrak{H}_0 * \mathfrak{P}(\lambda, \mathfrak{H})$ will also be symmetric if both h and h_0 are symmetric, and it will have a Gaussian component if $\beta_0 \neq 0$. As a consequence we will have $\alpha = 0$ from the symmetry, $\ell(x) = \lambda h(x) + \ell_0(x)$, and finally $\mathcal{L} = (0, \beta_0, \lambda h + \ell_0)$. The laws of the increments of the corresponding Lévy process will then be $\varphi(t) = \vartheta_0^{t/\tau} e^{\lambda t(\vartheta-1)/\tau}$, namely

$$\eta(u, t) = \frac{t}{\tau} \zeta_0(u) + \omega t [\vartheta(u) - 1]$$

so that the process will be the superposition of two independent processes: an \mathfrak{H}_0 –Lévy process plus a $\mathfrak{P}(\omega t, \mathfrak{H})$ compound Poisson process. Its trajectories will then be the paths of the \mathfrak{H}_0 –Lévy process, interspersed with Poisson random jumps with size law \mathfrak{H} . If then $h_0(x, t)$ is the *pdf* of $\vartheta_0^{t/\tau}(u)$, the t –increment *pdf*’s of our process will be

$$p(x, t) = e^{-\omega t} \sum_{k=0}^{\infty} \frac{(\omega t)^k}{k!} [h_0(t) * h^{*k}](x)$$

and the self–adjoint process generator

$$[Av](x) = \frac{\beta_0^2}{2} \partial_x^2 v(x) + \int_{y \neq 0} [v(x+y) - v(x)] [\lambda h(y) + \ell_0(y)] dy.$$

Possible examples of these \mathfrak{H}_0 are both the Gaussian and the non Gaussian stable laws (in particular the Cauchy process), and several self–decomposable laws as the Student or the Variance–Gamma. The relevant particular case of a Gaussian \mathfrak{H}_0 is discussed in the Section 2.4.

C Initial states

We define here a list of possible initial *pdf*'s and *wf*'s. To simplify our calculations we will choose the initial *pdf*'s to be centered and symmetric, and whenever convenient we will take pairs f_0, ψ_0 satisfying the relation $f_0 = |\psi_0|^2$. Remark that, while f_0 is a normalized (in L^1) *pdf* and φ_0 is a (non normalized, and possibly non normalizable) *chf*, ψ_0 and $\hat{\psi}_0$ must be both normalized (in L^2) *wf*'s so that we must always pay attention to the constants which are in front of them. Here moreover – to put in evidence the meaning of the involved quantities – our laws and time coordinates will be dimensional: the space a, b and time τ scaling parameters will be explicitly taken into account.

C.1 Normal \mathfrak{N}_b

Initial laws and *wf*'s with $f_0 = |\psi_0|^2$ are in this case

$$f_0(x) = \frac{e^{-x^2/2b^2}}{\sqrt{2\pi b^2}}, \quad \varphi_0(u) = e^{-b^2 u^2/2} \quad (35)$$

$$\psi_0(x) = \frac{e^{-x^2/4b^2}}{\sqrt[4]{2\pi b^2}}, \quad \hat{\psi}_0(u) = \sqrt[4]{\frac{2b^2}{\pi}} e^{-b^2 u^2} \quad (36)$$

Remark that, while ψ_0 is just the square root of f_0 , $\hat{\psi}_0$ is the *FT* of ψ_0 and its relation to φ_0 is given by the equation (15). The two *wf*'s, moreover, are both normalized in L^2 .

C.2 Cauchy $\mathfrak{C}_b = \mathfrak{T}_b(1)$

Initial laws and *wf*'s in this case are

$$f_0(x) = \frac{1}{b\pi} \frac{b^2}{b^2 + x^2}, \quad \varphi_0(u) = e^{-b|u|} \quad (37)$$

$$\psi_0(x) = \frac{1}{\sqrt{b\pi}} \sqrt{\frac{b^2}{b^2 + x^2}}, \quad \hat{\psi}_0(u) = \frac{\sqrt{2b}}{\pi} K_0(b|u|) \quad (38)$$

where K_0 is the modified Bessel function of order 0; it is easy to show indeed that (see for example [20] 9.6.21)

$$\hat{\psi}_0(u) = \frac{1}{\sqrt{b\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sqrt{\frac{b^2}{b^2 + x^2}} e^{-iux} dx = \frac{\sqrt{2b}}{\pi} K_0(b|u|).$$

The normalization $\|\hat{\psi}_0\|^2 = 1$, and the relation $\varphi_0 = \hat{\psi}_0 * \hat{\psi}_0$, are then

$$\begin{aligned} \int_{u \neq 0} K_0^2(b|u|) du &= \frac{\pi^2}{2b} \\ \int_{v \neq 0, u} K_0(b|u-v|) K_0(b|v|) dv &= \frac{\pi^2}{2b} e^{-b|u|} \end{aligned}$$

The first can be reduced to

$$\int_0^{+\infty} K_0^2(u) du = \frac{\pi^2}{4}$$

which can be verified by direct calculation. On the other hand the convolution, that can be reduced to the dimensionless relation

$$\int_{v \neq 0, u} K_0(|u-v|) K_0(|v|) dv = \frac{\pi^2}{2} e^{-|u|},$$

does not seem to be an otherwise known result.

C.3 3–Student $\mathfrak{T}_b(3)$

Initial laws and *wf*'s in this case are

$$f_0(x) = \frac{2}{b\pi} \left(\frac{b^2}{b^2 + x^2} \right)^2, \quad \varphi_0(u) = e^{-b|u|}(1 + b|u|) \quad (39)$$

$$\psi_0(x) = \sqrt{\frac{2}{b\pi}} \frac{b^2}{b^2 + x^2}, \quad \hat{\psi}_0(u) = \sqrt{b} e^{-b|u|} \quad (40)$$

It is very easy to show that $\hat{\psi}_0$ is the right *FT* of ψ_0

$$\hat{\psi}_0(u) = \sqrt{\frac{2}{b\pi}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{b^2}{b^2 + x^2} e^{-iux} dx = \sqrt{b} e^{-b|u|}$$

while here again an elementary calculation shows also that $\varphi_0 = \hat{\psi}_0 * \hat{\psi}_0$.

C.4 Variance–Gamma $\mathfrak{VG}_b(\nu)$

In the general Variance–Gamma case, to make calculations possible, we will not always choose pairs of initial *pdf*'s and *wf*'s satisfying $\psi_0 = \sqrt{f_0}$. A possible example then is

$$f_0(x) = \frac{2}{2^\nu \Gamma(\nu) \sqrt{2\pi} b} \left(\frac{|x|}{b} \right)^{\nu-\frac{1}{2}} K_{\nu-\frac{1}{2}} \left(\frac{|x|}{b} \right), \quad \varphi_0(u) = \left(\frac{1}{1 + b^2 u^2} \right)^\nu \quad (41)$$

$$\psi_0(x) = \sqrt{\frac{2\Gamma(\nu + \frac{1}{2})}{b\pi\Gamma(\nu)\Gamma(2\nu - \frac{1}{2})}} \left(\frac{|x|}{b} \right)^{\nu-\frac{1}{2}} K_{\nu-\frac{1}{2}} \left(\frac{|x|}{b} \right), \quad (42)$$

$$\hat{\psi}_0(u) = \sqrt{\frac{b\Gamma(2\nu)}{\sqrt{\pi}\Gamma(2\nu - \frac{1}{2})}} \left(\frac{1}{1 + b^2 u^2} \right)^\nu \quad (43)$$

where the functions are chosen in order to have an evolution easy to calculate. The *wf*'s ψ_0 and $\hat{\psi}_0$, in any case, are both normalized in L^2 (as can be seen by direct

calculation) and are apparently in the FT relation. As a consequence here the Lévy and the L - S evolutions will possibly start with different pdf 's. In fact the usual relation $f_0 = |\psi_0|^2$ could be easily restored just in the particular case of $\nu = 1$, namely for an initial Laplace law $\mathfrak{L}_b = \mathfrak{VG}_b(1)$:

$$f_0(x) = \frac{e^{-|x|/b}}{2b}, \quad \varphi_0(u) = \frac{1}{1 + b^2 u^2} \quad (44)$$

$$\psi_0(x) = \frac{e^{-|x|/2b}}{\sqrt{2b}}, \quad \hat{\psi}_0(u) = \sqrt{\frac{b}{\pi}} \frac{2}{1 + 4b^2 u^2} \quad (45)$$

Here it is elementary to check indeed that $\psi_0 = \sqrt{f_0}$, that $\hat{\psi}_0$ is the FT of ψ_0 , and finally that $\varphi_0 = \hat{\psi}_0 * \hat{\psi}_0$. This particular case, however, is not really easier than the general case of the Variance–Gamma process. In fact, as we will see soon, the parameter affected by the time evolution is exactly ν , so that it is of no help to start with $\nu = 1$ if it immediately becomes $\nu \neq 1$.

C.5 Relativistic qm $\mathfrak{R}_b(\nu)$

Again to make calculations easy we will choose as initial chf and wf FT respectively

$$f_0(x) = \frac{\nu e^\nu K_1\left(\sqrt{\nu^2 + x^2/b^2}\right)}{b\pi\sqrt{\nu^2 + x^2/b^2}}, \quad \varphi_0(u) = e^{\nu(1 - \sqrt{1 + b^2 u^2})} \quad (46)$$

$$\psi_0(x) = \frac{\nu K_1\left(\sqrt{\nu^2 + x^2/b^2}\right)}{\sqrt{\pi b K_1(2\nu)(\nu^2 + x^2/b^2)}}, \quad \hat{\psi}_0(u) = \sqrt{\frac{b}{2K_1(2\nu)}} e^{-\nu\sqrt{1 + b^2 u^2}} \quad (47)$$

which are in a relation similar to that of (41)–(43). It is easy to recognize that the wf 's are correctly normalized in L^2 .

D Transition laws and propagators

We will list here a few examples of background Lévy noises by paying attention to pick up processes with a known transition pdf associated to the evolution equation (13) and a known propagator associated to the free L - S equation (17).

D.1 Normal $\mathfrak{N}(2Dt)$

Here the background noise is a Wiener process: take a \mathfrak{N}_a law with Lévy triplet $\mathcal{L} = (0, a, 0)$

$$f(x) = \frac{e^{-x^2/2a^2}}{\sqrt{2\pi a^2}}, \quad \varphi(u) = e^{-a^2 u^2/2}$$

The transition law of the corresponding Lévy process is then $\mathfrak{N}(2Dt)$ with $D = a^2/2\tau$, namely

$$q(x, t) = \frac{e^{-x^2/4Dt}}{\sqrt{4\pi Dt}}, \quad \chi(u, t) = e^{-Dtu^2} \quad (48)$$

and the *pdf* evolution equation (13) is the usual Fokker–Planck equation

$$\partial_t p(x, t) = D \partial_x^2 p(x, t)$$

The corresponding *L-S* propagator $\mathfrak{N}(2iDt)$ is again formally normal albeit with an imaginary variance:

$$g(x, t) = \frac{e^{-x^2/4iDt}}{\sqrt{4\pi iDt}}, \quad \gamma(u, t) = e^{-iDtu^2} \quad (49)$$

and hence the *L-S* equation (17) is the usual free Schrödinger equation

$$i\partial_t \psi(x, t) = -D \partial_x^2 \psi(x, t).$$

D.2 Cauchy \mathfrak{C}_{ct}

From the Cauchy law \mathfrak{C}_a , a typical stable, non Gaussian law with Lévy triplet $\mathcal{L} = (0, 0, a/\pi x^2)$ and with

$$f(x) = \frac{1}{a\pi} \frac{a^2}{a^2 + x^2}, \quad \varphi(u) = e^{-a|u|}$$

we get the transition law \mathfrak{C}_{ct} of the Cauchy process with $c = a/\tau$:

$$q(x, t) = \frac{1}{\pi ct} \frac{c^2 t^2}{c^2 t^2 + x^2}, \quad \chi(u, t) = e^{-ct|u|} \quad (50)$$

and the corresponding process equation (13)

$$\partial_t p(x, t) = \int_{y \neq 0} [p(x + y, t) - p(x, t)] \frac{c}{\pi y^2} dy. \quad (51)$$

On the other hand the *L-S* propagator \mathfrak{C}_{ict} is

$$g(x, t) = \frac{1}{i\pi} \frac{ct}{c^2 t^2 - x^2}, \quad \gamma(u, t) = e^{-ict|u|}. \quad (52)$$

and the *L-S* equation (17)

$$i\partial_t \psi(x, t) = - \int_{y \neq 0} [\psi(x + y, t) - \psi(x, t)] \frac{c}{\pi y^2} dy \quad (53)$$

Remark that, at variance with the transition *pdf* (50), the Cauchy–Schrödinger propagator (52) has two simple poles in $x = \pm ct$ drifting away from the center $x = 0$ with velocity c .

D.3 Variance–Gamma $\mathfrak{VG}_a(\omega t)$

Take a *sd*, non stable Variance–Gamma law $\mathfrak{VG}_a(\lambda)$ with symmetric Lévy triplet $\mathcal{L} = (0, 0, \lambda e^{-|x|/a}/|x|)$ and with

$$f(x) = \frac{2}{2^\lambda \Gamma(\lambda) \sqrt{2\pi} a} \left(\frac{|x|}{a} \right)^{\lambda - \frac{1}{2}} K_{\lambda - \frac{1}{2}} \left(\frac{|x|}{a} \right), \quad \varphi(u) = \left(\frac{1}{1 + a^2 u^2} \right)^\lambda$$

The transition law will then be $\mathfrak{VG}_a(\omega t)$ with $\omega = \lambda/\tau$:

$$q(x, t) = \frac{2}{2^{\omega t} \Gamma(\omega t) \sqrt{2\pi} a} \left(\frac{|x|}{a} \right)^{\omega t - \frac{1}{2}} K_{\omega t - \frac{1}{2}} \left(\frac{|x|}{a} \right), \quad \chi(u, t) = \left(\frac{1}{1 + a^2 u^2} \right)^{\omega t} \quad (54)$$

and the corresponding process equation (13)

$$\partial_t p(x, t) = \omega \int_{y \neq 0} [p(x + y, t) - p(x, t)] \frac{e^{-|y|/a}}{|y|} dy.$$

so that the evolution will only affect the parameter λ , while a will always be the same. Then for the *L-S* propagator $\mathfrak{VG}_a(i\omega t)$ we have

$$g(x, t) = \frac{2}{2^{i\omega t} \Gamma(i\omega t) \sqrt{2\pi} a} \left(\frac{|x|}{a} \right)^{i\omega t - \frac{1}{2}} K_{i\omega t - \frac{1}{2}} \left(\frac{|x|}{a} \right), \quad \gamma(u, t) = \left(\frac{1}{1 + a^2 u^2} \right)^{i\omega t} \quad (55)$$

while the *L-S* equation (17) becomes

$$i\partial_t \psi(x, t) = -\omega \int_{y \neq 0} [\psi(x + y, t) - \psi(x, t)] \frac{e^{-|y|/a}}{|y|} dy$$

D.4 Wiener–Poisson $\mathfrak{N}(2Dt) * \mathfrak{P}(\omega t, \mathfrak{N})$

We will consider here two examples of *id*, non *sd* background noise: for notations and details see Section 2.4 and Appendix B. Take first the law $\mathfrak{N}_\sigma * \mathfrak{P}(\lambda, \mathfrak{N}_a)$ discussed in the Section 2.4. From its *chf* we see that, with $\omega = \lambda/\tau$ and $D = \sigma^2/2\tau$, the transition law $\mathfrak{N}(2Dt) * \mathfrak{P}(\omega t, \mathfrak{N}_a)$

$$q(x, t) = e^{-\omega t} \sum_{k=0}^{\infty} \frac{(\omega t)^k}{k!} \frac{e^{-x^2/2(ka^2 + 2Dt)}}{\sqrt{2\pi(ka^2 + 2Dt)}}, \quad \chi(u, t) = e^{\omega t(e^{-a^2 u^2/2} - 1)} e^{-Dtu^2} \quad (56)$$

The corresponding Wiener–Poisson process will have sample paths which are Brownian trajectories interspersed with Gaussian jumps at Poisson times with intensity λ . The process *pdf*'s then have an elementary form as time dependent Poisson mixtures of time dependent normal laws and the corresponding process equation (13) will become

$$\partial_t p(x, t) = D \partial_x^2 p(x, t) + \omega \int_{-\infty}^{+\infty} [p(x + y, t) - p(x, t)] \frac{e^{-y^2/2a^2}}{\sqrt{2\pi a^2}} dy.$$

The L - S propagator $\mathfrak{N}(2iDt) * \mathfrak{P}(i\omega t, \mathfrak{N}_a)$ now is

$$g(x, t) = e^{-i\omega t} \sum_{k=0}^{\infty} \frac{(i\omega t)^k}{k!} \frac{e^{-x^2/2(ka^2+2iDt)}}{\sqrt{2\pi(ka^2+2iDt)}}, \quad \gamma(u, t) = e^{i\omega t(e^{-a^2u^2/2}-1)} e^{-iDtu^2} \quad (57)$$

and the L - S equation (17) becomes

$$i\partial_t \psi(x, t) = -D\partial_x^2 \psi(x, t) - \omega \int_{-\infty}^{+\infty} [\psi(x+y, t) - \psi(x, t)] \frac{e^{-y^2/2a^2}}{\sqrt{2\pi a^2}} dy.$$

As a second example take the law $\mathfrak{N}_\sigma * \mathfrak{P}(\lambda, \mathfrak{D}_a)$ discussed in the Section 2.4: from its lch $\eta(u, t) = \omega t(\cos au - 1) - Dtu^2$ we see that the law of the corresponding Lévy process is $\mathfrak{N}(2Dt) * \mathfrak{P}(\omega t, \mathfrak{D}_a)$ and hence

$$\begin{aligned} q(x, t) &= e^{-\omega t} \sum_{k=0}^{\infty} \frac{(\omega t)^k}{k!} \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} \frac{e^{-[x-(k-2j)a]^2/4Dt}}{\sqrt{4\pi Dt}} \\ \chi(u, t) &= e^{\omega t(\cos au - 1) - Dtu^2} \end{aligned} \quad (58)$$

while the process equation (13) is

$$\partial_t p(x, t) = D\partial_x^2 p(x, t) + \omega \frac{p(x+a, t) - 2p(x, t) + p(x-a)}{2}.$$

This process will have sample paths which are again Brownian trajectories interspersed with jumps $\pm a$ at Poisson times with intensity λ . The L - S propagator $\mathfrak{N}(2iDt) * \mathfrak{P}(i\omega t, \mathfrak{D}_a)$ instead is

$$\begin{aligned} g(x, t) &= e^{-i\omega t} \sum_{k=0}^{\infty} \frac{(i\omega t)^k}{k!} \frac{1}{2^k} \sum_{j=0}^k \binom{k}{j} \frac{e^{-[x-(k-2j)a]^2/4iDt}}{\sqrt{4\pi iDt}} \\ \gamma(u, t) &= e^{i\omega t(\cos au - 1) - iDtu^2} \end{aligned} \quad (59)$$

and the L - S equation (17) is

$$i\partial_t \psi(x, t) = -D\partial_x^2 \psi(x, t) - \omega \frac{\psi(x+a, t) - 2\psi(x, t) + \psi(x-a)}{2}.$$

D.5 Relativistic qm $\mathfrak{R}_a(\omega t)$

We immediately see from the chf of $\mathfrak{R}_a(\lambda)$ that the corresponding Lévy process $\mathfrak{R}_a(\omega t)$ will have as transition law

$$q(x, t) = \frac{\omega t e^{\omega t} K_1\left(\sqrt{\omega^2 t^2 + x^2/a^2}\right)}{\pi a \sqrt{\omega^2 t^2 + x^2/a^2}}, \quad \chi(u, t) = e^{\omega t(1 - \sqrt{1+a^2u^2})} \quad (60)$$

with $\omega = \lambda/\tau$ as usual. We can also explicitly write the process equation (13)

$$\partial_t p(x, t) = \omega \int_{y \neq 0} [p(x + y, t) - p(x, t)] \frac{K_1(|y|/a)}{\pi|y|} dy.$$

On the other hand the L - S propagator $\mathfrak{R}_a(i\omega t)$ will be given by

$$g(x, t) = \frac{i\omega t e^{i\omega t} K_1 \left(\sqrt{-\omega^2 t^2 + x^2/a^2} \right)}{\pi a \sqrt{-\omega^2 t^2 + x^2/a^2}}, \quad \gamma(u, t) = e^{i\omega t(1 - \sqrt{1 + a^2 u^2})} \quad (61)$$

with singularities in $x = \pm a\omega t$, and corresponds to the L - S equation

$$i\partial_t \psi(x, t) = -\omega \int_{y \neq 0} [\psi(x + y, t) - \psi(x, t)] \frac{K_1(|y|/a)}{\pi|y|} dy.$$

We remember here, as remarked in the Section 2.5, that this essentially is the integro–differential form of the well known relativistic, free Schrödinger equation

$$i\hbar \partial_t \psi(x, t) = \sqrt{m^2 c^4 - c^2 \hbar^2 \partial_x^2} \psi(x, t)$$

that we recover by taking $\omega = \lambda/\tau = mc^2/\hbar$, $a = \hbar/mc$, and by reabsorbing an irrelevant constant term mc^2 in a phase factor of the wf [1].

E Second kind Beta laws

If Z is a rv with a (dimensionless) *Beta law* $\mathfrak{B}(\alpha, \beta)$ ($\alpha, \beta > 0$) namely with *pdf*

$$f_Z(z) = \frac{z^{\alpha-1}(1-z)^{\beta-1}}{B(\alpha, \beta)}, \quad 0 \leq z \leq 1$$

then $Y = Z/(1-Z)$ is distributed according to a *second kind Beta law* $\tilde{\mathfrak{B}}(\alpha, \beta)$ with *pdf* (see for example [21])

$$f_Y(y) = \frac{1}{B(\alpha, \beta)} \frac{y^{\alpha-1}}{(1+y)^{\alpha+\beta}}, \quad 0 \leq y.$$

We could also introduce a scale parameter a to get the types $\mathfrak{B}_a(\alpha, \beta)$ and $\tilde{\mathfrak{B}}_a(\alpha, \beta)$, but to simplify the notation we will first consider only the dimensionless laws. Take now a third rv $X = \epsilon\sqrt{Y}$ where \sqrt{Y} is the *positive* square root of Y , while ϵ is another independent rv taking the two values ± 1 with the same probability $1/2$. We find then that its *pdf* is

$$f_X(x) = \frac{1}{B(\alpha, \beta)} \frac{(x^2)^{\alpha-\frac{1}{2}}}{(1+x^2)^{\alpha+\beta}}.$$

We will use for these laws the symbol $\tilde{\mathfrak{B}}^{1/2}(\alpha, \beta)$ because X is the square root of a second kind Beta *rv*. In particular we recover the family of the Student laws as

$$\tilde{\mathfrak{B}}^{1/2}\left(\frac{1}{2}, \frac{\lambda}{2}\right) = \mathfrak{T}(\lambda), \quad \lambda > 0,$$

while $\tilde{\mathfrak{B}}^{1/2}(3/2, 1/2)$ is the law introduced in the Section 4.2 to describe the evolution of an initial Student law $\mathfrak{T}(3)$ by a Cauchy transition *pdf*. For this law we have

$$f(x) = \frac{2}{\pi} \frac{x^2}{(1+x^2)^2}, \quad \varphi(u) = (1-|u|)e^{-|u|},$$

and (as for the usual Cauchy laws) we find that it has neither an expectation, nor a finite variance. The decomposition (20) could now be written also as a relation within the (dimensional) family $\tilde{\mathfrak{B}}_a^{1/2}(\alpha, \beta)$ by remembering that $\mathfrak{C}_{ct} = \tilde{\mathfrak{B}}_{ct}^{1/2}(1/2, 1/2)$ and $\mathfrak{T}_b = \tilde{\mathfrak{B}}_b^{1/2}(1/2, 3/2)$. In fact, for given arbitrary scale parameters a and b , and with

$$\begin{aligned} P &= \frac{1}{2} \frac{a}{a+b} \\ Q &= \frac{1}{2} \frac{a+2b}{a+b} = \frac{1}{2} \left(1 + \frac{b}{a+b}\right), \end{aligned}$$

we easily see that (20) is a special case (for $a = ct$) of

$$\tilde{\mathfrak{B}}_a^{1/2}\left(\frac{1}{2}, \frac{1}{2}\right) * \tilde{\mathfrak{B}}_b^{1/2}\left(\frac{1}{2}, \frac{3}{2}\right) = P \tilde{\mathfrak{B}}_{a+b}^{1/2}\left(\frac{3}{2}, \frac{1}{2}\right) + Q \tilde{\mathfrak{B}}_{a+b}^{1/2}\left(\frac{1}{2}, \frac{3}{2}\right)$$

namely

$$\begin{aligned} (1+b|u|)e^{-(a+b)|u|} &= P[1-(a+b)|u|]e^{-(a+b)|u|} + Q[1+(a+b)|u|]e^{-(a+b)|u|} \\ \frac{(a+b)^2(a+2b)+ax^2}{\pi[(a+b)^2+x^2]^2} &= \frac{2P}{\pi} \frac{(a+b)x^2}{[(a+b)^2+x^2]^2} + \frac{2Q}{\pi} \frac{(a+b)^3}{[(a+b)^2+x^2]^2} \end{aligned}$$

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